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ABSTRACT

The implementation of a simple torsion pendulum is presented in order to obtain the inertia parameters of an irregular-shaped body. The system consists of a fixed frame and an inertial frame which is attached by two strings to the fixed frame. The idea of the torsional pendulum is proposed in the book "Lagrangian Dynamics" [1], following the idea presented in it an apparatus was created in order to measure the inertial moments of an aquatic vehicle along its yaw axis of symmetry. In order to make said measurements it is necessary to determine the torsion constant C of the string used, steel caliber 12, and the inertial moment of the inertial frame I_F for which two measurements were made using a homogeneous body whose inertia moments where known (homogeneous aluminum cylinder). For calibration purposes the equation for the inertia moments was obtained for a sphere composed of two different materials (basketball), using said expression theoretical calculations were made and the apparatus was used to experimentally measure the moments of inertia, the results compared were found to be similar. Finally, the yaw moment of inertia for the mini submarine was measured.

Key Words: *Moments of inertia, torsional pendulum, center of mass, irregular-shaped body.*

1. Introduction

The modeling of mechanical systems composed of many bodies is a fundamental part in the design and construction for the classical and modern control area of study [2]. One of the fundamental parts for the modeling of a multibody system are the coordinates of the center of mass, and the inertia matrix [3]. Methods for finding the inertia matrix, also known as the inertia tensor, are given mathematically in many physics courses, however these methods only work for systems or bodies who are symmetrical and homogeneous [4]. It is cases where the body or system is not symmetrical nor homogeneous where problems arise and conventional methods become obsolete forcing the use of different alternative experimental methods. A software often used in the aid of mechanical structural analysis is Solidworks [5], this program allows the user to design any structure as well as choose the material and it automatically calculates the numerical values for the center of mass and the elements which make up the inertia matrix with respect to the systems symmetrical axes, nevertheless, this method provides an estimation of the real values [6].

Various procedures exist in order to obtain the inertia moments, much of which have already been implemented. There are two principal types of systems: those that use acceleration and those that use oscillations. In the latter group lie the torsional pendulum and spring systems [7]. The pendulum systems have an advantage in security and simplicity when it comes to their construction [8]. One of these systems is the trifilar pendulum with which inertia parameters can be found, specifically the moments of inertia [6,9-10]. The trifilar pendulums have some disadvantages which can be minimized using diverse approaches in their construction methods: making use of load cells and balancing weights in order to eliminate the need to know or calculate the center of gravity [9], using a universal joint and a tri-coordinate measuring machine [6]. A setback in the implementation of a trifilar pendulum system is the equipment cost, since these type of systems are mostly used in industrial settings where they also have a skilled set of workers [8]. It is very hard to build such a complex systems in a university where such equipment and resources are seldom available, and therefore ingenious cost-effective methods have to be implemented.

The torsion pendulum presented in this article was the one used to measure and calculate the elements of the inertia matrix. This method has the advantages of a trifilar pendulum as well as a considerable reduction in construction costs [6].

The structural composition of the tension pendulum is as follows: a fixed frame which houses a smaller frame, named inertial frame; the inertial frame is suspended by two steel strings attached to the upper and lower sides of the fixed frame, creating the torsional pendulum. The body whose inertia parameters want to be acquired is placed within the inertial frame.

There are two stages needed prior to measuring the elements which make up the inertia matrix, which is the whole purpose of the system. The first stage consists in utilizing symmetrical bodies whose inertia moments are known theoretically and using them in order determine the two unknown parameters of our system, which are the string's torsion constant and the inertial moment of the inertial frame, this is very important since these unknowns are found in our equation. The second stage consists of verifying that the numerical values measured in the previous stage for the unknowns are correct, this is done by using another geometrically simple body whose calculation of its theoretical moments of inertia are easy to find, then using the torsional pendulum the experimental values are found for said body and the results are then compared to see if their similarity is satisfactory, this way the system can be deemed calibrated and the measuring of the inertia tensor can be pursued for a much more complex system or body with the guarantee that the resulting measurements can be trusted.

The work is organized as follows: in section 2 the theoretical fundaments on which the system is based are presented, these are the fundamental equations for the rigid-body analysis for a moving body using the Newton-Euler method, theory and examples about the inertia matrix and other important expressions regarding the torsional pendulum. In section 3 the experimental process of the system is outlined, starting from its construction, then the process in which the system is calibrated for its proper use, and lastly the measurements made and results obtained. The last section in this article discusses the results obtained, focusing on the advantages and disadvantages of the system created as well as the conclusion.

2. Mathematical basis

In order to describe the movement of a rigid-body, equations of motion are required, these can be obtained from the following expression:

$$\boldsymbol{F_R} = m\, \boldsymbol{a_{CM}}. \tag{1}$$

This equation comes from Newton's second law of motion, where $\boldsymbol{F_R}$ is the sum of the forces that act on a mass$m$ and$\boldsymbol{a_{CM}}$ is the acceleration of the center of mass.

$$\boldsymbol{H_o} = \widehat{I_o} \cdot \boldsymbol{\alpha}. \tag{2}$$

This is Euler's equation, where $\boldsymbol{H_o}$ is the sum of the torque that act on a body with respect to an origin O. $\widehat{I_o}$ is the inertia tensor and $\boldsymbol{\alpha}$ is the angular acceleration of the body.

With respect to equations (1) and (2) it is obvious that the center of mass and the inertia tensor of the body have to be known in order to be able to analyze the movement of the system. This implies that the geometrical form of our body has to be considered in order to describe the rotational movement of the rigid-body. The geometric form of a rotating object can be seen in the object's resist to rotational acceleration, and it is the rotational analog to mass.

The physical quantity where the inertia to rotate appears is the inertia tensor. In order to visualize this it is necessary to take a note that the angular momentum of a rigid body is given by:

$$\boldsymbol{M} = \hat{I} \cdot \boldsymbol{\omega}, \tag{3}$$

where $\hat{I}$ is the inertia tensor, and $\boldsymbol{\omega}$ the angular velocity.

The inertia tensor for a body in general is as follows:

$$\hat{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}, \tag{4}$$

where

$$I_{ij} = \int_v^V \rho(r)(\delta_{ij} r^2 - r_i r_j)\, dV, \tag{5}$$

With $\rho(r)$being the density of the mass, $V$the volume of the body,$r^2 = x^2 + y^2 + z^2$and the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & sii \neq j \\ 1 & sii = j. \end{cases}$$

This quantity depends on the chosen coordinate system. If the origin of said coordinate system coincides with the center of mass and the axes are also along the symmetry axes then the inertia tensor is said to be diagonal.

For a solid cylinder, the inertia tensor is given by:

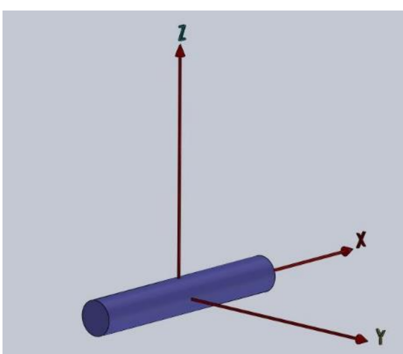

Fig. 1 Solid cylinder in a Cartesian reference system

$$\hat{I}_c = \begin{pmatrix} \frac{1}{2} m R^2 & 0 & 0 \\ 0 & \frac{1}{2} m (R^2 + L^2) & 0 \\ 0 & 0 & \frac{1}{2} m (R^2 + L^2) \end{pmatrix} \tag{6}$$

where R is the radius of the cylinder, L the longitude and m the mass.

For a homogeneous sphere, the inertia tensor is given by:

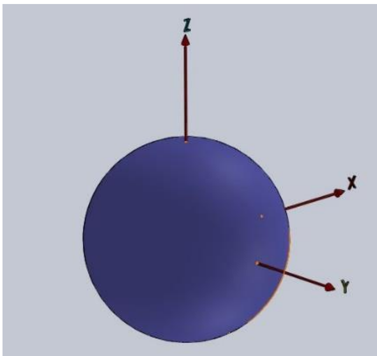

Fig. 2 Solid sphere in a Cartesian reference system

$$\hat{I}_e = \begin{pmatrix} \frac{2}{5}mr^2 & 0 & 0 \\ 0 & \frac{2}{5}mr^2 & 0 \\ 0 & 0 & \frac{2}{5}mr^2 \end{pmatrix} \quad (7)$$

where r is the radius of the sphere and m the mass.

The inertia tensor was then calculated for a nonhomogeneous sphere, as shown in Fig. 3. The inertia tensor for the sphere, which is composed of two different materials is given by:

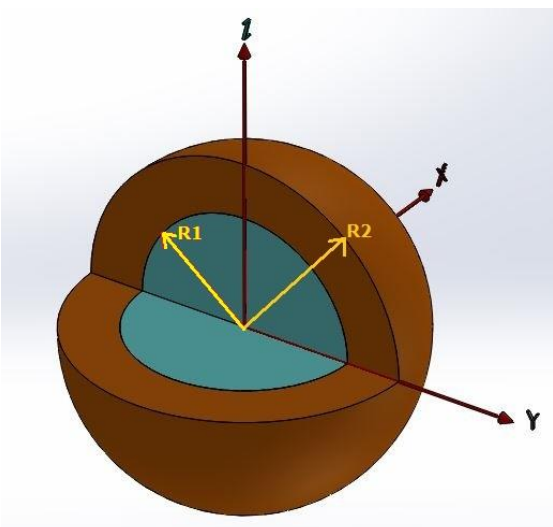

Fig. 3 Nonhomogeneous sphere in a Cartesian reference system

$$\hat{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix},$$

where

$$I_{xx} = I_{yy} = I_{zz} = I_{en}.$$

The inertia moment is obtained by using equation (5),

$$I_{en} = \frac{8}{3}\pi\left[\left(\frac{M_1}{V_1} - \frac{M_2}{V_2}\right)\frac{{R_1}^5}{5} + \frac{M_2}{V_2}\frac{{R_2}^5}{5}\right], \quad (8)$$

where $V_1 = \frac{4}{3}\pi R_1{}^3$; $V_2 = \frac{4}{3}\pi\left(R_2{}^3 - R_1{}^3\right)$ are the volumes of both materials, with M_1being the mass of material 1, and M_2 the mass of material 2.

In this case it can be noted that if$R_1 = 0\ and R_2 = R$, it would be the same as a homogeneous sphere, since$V_1 = 0$, $V_2 = \frac{4}{3}\pi R^3$ y $M_2 = M$.Which takes us to$I_{en} = \frac{2}{5}MR^2$, which would mean that it is indeed the same expression as that for a homogeneous sphere.

2.1 Torsional pendulum

The torsional pendulum is a system that consists of a rigid-body suspended by means of a string from a fixed support.

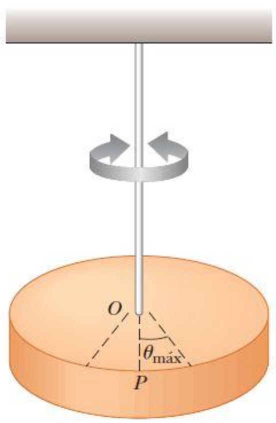

Fig. 4 General form of a torsional pendulum

When the object rotates a given angle θ, the string provides the restoring force which is proportional to its angular position, meaning,

$$\tau = -C_0\theta,$$

Where C_0 is the restoring constant and τ is the restoring torque. This gives us that the total torque applied to the system, considering friction, is:

$$\boldsymbol{H} = \left(C_1\dot{\theta} - C_0\theta\right)\hat{k} \tag{9}$$

where $C_1\dot{\theta}$ is the torque applied by the friction. Applying Euler's law from equation (2) we have$\boldsymbol{H} = \hat{I}\cdot\boldsymbol{\alpha}$, where

$$\boldsymbol{\alpha} = \ddot{\theta}\hat{k} \tag{10}$$

Substituting (9) and (10) in Euler's equation

$$(C_1\dot{\theta} - C_0\theta)\hat{k} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \ddot{\theta} \end{pmatrix} = I_{zz}\ddot{\theta}\hat{k} \qquad (11)$$

resulting in

$$C_1\dot{\theta} - C_0\theta = I_{zz}\ddot{\theta} \qquad (12)$$

By considering $C_1\dot{\theta}$ very small, equation (12) can be expressed as a homogeneous differential equation

$$\ddot{\theta} + {\omega_0}^2\theta = 0 \,, \qquad (13)$$

where

$$\omega_0 = \sqrt{C_0/I_{zz}}, \qquad (14)$$

Solving for theta

$$\theta = \theta_0 \operatorname{sen}(\omega_0 t + \gamma) \qquad (15)$$

By definition the angular frequency is as follows:

$$\omega_0 = 2\pi f = \frac{2\pi}{P}$$

where we get period as follows:

$$P = 2\pi\frac{1}{\omega_0} \qquad (16)$$

and the period in terms of the inertia tensor

$$P = 2\pi\sqrt{I_{zz}/C_0} \qquad (17)$$

It is thanks to this equation that, by knowing the period P and the torsion constant C_0, we are able to find the inertia moment I_{zz}.

3 **Experimental procedure**

3.1 Construction

The system consists of three main parts, and external frame which serves as the support, where an inner frame, named inertial frame, is suspended by string, this acts as the torsional pendulum, a pair of threaded bar and nuts which are used to fix the body within the inertial frame.

The external frame is rectangular shaped, it is made from structural tubing, and it has the following dimensions: height 149.4cm and width 83.5cm. it has two supports on the bottom, which help in providing stability to the frame. There are two holes, with diameter D, located on the upper and lower parts of the frame, which is where a string runs through and fixes it with the inertial frame. On the upper part of the frame, a tightening-system was coupled which is used to tighten the upper string, which unites it to the inertial frame.

The inertial frame has a rectangular shape, it’s made from structural tubing, and has the following dimensions: height 75.3cm and width 50.2cm. it has four holes, one on each side; the string runs through the upper and lower holes connecting the inertial frame to the external frame, the threaded bars run through the holes on the sides of the inertial frame and are fixed in place with the nuts. It is very important to take into account the mass of the inertial frame when using the torsional pendulum since the moment we obtain is the sum of the inertial frame moment, and that of the threaded bars, and that of the object we are analyzing. Mathematically, through the equations that were acquired previously, we can obtain the independent moment of our target-body, and it is here that we can see the influence the mass of the inertial frame, since the moment of the frame depends solely on its mass and shape, which is why for accuracy purposes it is highly recommended that the mass of the inertial frame be kept as small as possible since a massive frame would make the moment of the body we are trying to acquire insignificant. It was with this in mind that the materials where chosen, and just as important the measured mass of the frame which is of 2.453Kg as well as the mass of the threaded bar and the nuts used to fix the target-body within the frame, which is of 0.214Kg.

The strings that connect the inertial frame to the external frame, which together make up the torsional pendulum, are made from steel caliber 12 and have a plastic housing. This string was selected due to its characteristics since it has to be able to withstand the combined weight of the inertial frame and the target-body and is therefore exposed to high tensions. In order to determine the right string used for our torsional pendulum many tests were conducted with different types of strings, weeding out those that broke due to the tension and those that were too stiff to be able to properly tighten.

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Fig. 5 External Frame, inertial frame and string (full system)

3.2 Calibration

Once the apparatus is ready, going back to equation (17) the moment I_{zz} is substituted as the sum of the moments $I_i + I_F$ where I_F is the inertial frames inertia moment, $I_i$is the moment to measure of our body and$i$ is the axis (x, y, z). $C_0$is replaced by C for the torsion constant of the strings and t for the oscillating period P. This is the resulting expression from which all of our calculations will be based from,

$$P = 2\pi\sqrt{\frac{(I_i+I_F)}{C}}. \quad (18)$$

It is necessary to find the torsion constant and the inertial frame's inertial moment, in order to do so some measurements and calculations were necessary.

The first step is to take a homogeneous body or object that is symmetrical in shape, the chosen object for this experiment was an aluminum cylinder whose characteristics are shown in table 1.

Physical Characteristics	**Data**
L-Length	41.4 cm
R-Radius	2.2 cm
M-Mass	1.714 kg

Table 1 Cylinder characteristics

The cylinder's data was substituted into matrix (6) in order to obtain the theoretical inertia moments for the cylinder. These are shown in table 2.

Inertia Moment	Data
$I_x{}^C$	$0.00041 kgm^2$
$I_y{}^C = I_z{}^C$	$0.02519 kgm^2$

Table 2 Cylinder'sinertia moments

From equation (18), the unknowns are I_F and C, and for this reason it is necessary to have 2 equations, in other words we need to different values for our period$P$ as well as two different values for$I_i$giving us a system of equations. Utilizing the aluminum cylinder, two values for I_i are obtained (taking different axes), via experimental measurement the period P is obtained, using this data it was easy to construct our system of equations that would in turn give us the value of our unknowns.

The measuring of the oscillation periods was then done for the cylinder for each particular position. The oscillation period was obtained by measuring the time it took the inertial frame, with the cylinder inside, to go back to its starting point after being let go from a given angular position. The period $P_1$was taken by placing the cylinder in its x-axis, along the pendulums rotational axis, that isI_x, period P_2 was taken by placing the cylinder in its y-axis along the pendulums rotational axis. The way the cylinders axes were taken are shown in Fig. 1.


Fig. 6 Measurement taken for the cylinder's x-axis

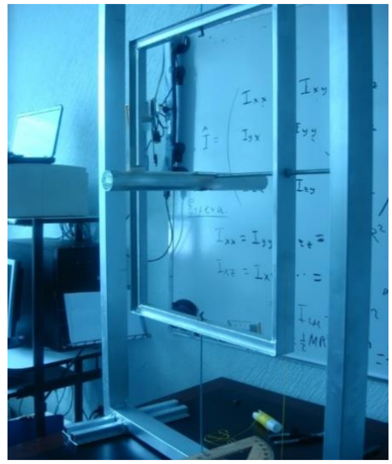
Fig. 7 Measurement taken for the cylinder's y-axis

The results of the measured oscillation periods and their corresponding moments are shown in table 3.

Moment	Period	Period
$I_x{}^C$	P_1	$3.51s$
$I_y{}^C$	P_2	$3.67s$

Table 3 Oscillation periods for the given moments

Taking into consideration equation (18), by inputting the cylinder's moments and the data acquired from the measurements we get the following

$$P_1 = 2\pi\sqrt{\frac{(I_x{}^C + I_F)}{C}} \quad (19)$$

$$P_2 = 2\pi\sqrt{\frac{(I_y{}^C + I_F)}{C}} \quad (20)$$

Equations (19) and (20) were solved simultaneously, utilizing the numerical values from tables 1 and 2, for the unknowns I_F and C.

The results calculated are shown in table 4.

I_F	$0.2653 kgm^2$
C	$0.8517\, kgm^2/s^2$

Table 4 Built torsional pendulum's parameters, see Fig. 5

Now that the needed parameters are known, the system can be used to measure the moments of inertia for any body.

The next step was to verify that the system would provide correct results, for this it was necessary to measure the inertia moment of another object, theoretically as well as experimentally, and certify that both values are alike.

A basketball was used, which can be seen as a nonhomogeneous sphere made up of two different materials as seen in Fig. 3 where material 1 is the air in the ball and material 2 is the rubber from which the ball is made. In order to calculate the inertia moment equation (8) was used, which called for the mass and diameter of the ball. This data is shown in table 5. Once again, from equation (8) the inertia moment for the ball is:

	Data
Mass of air - M_1	6g
Mass of the ball's rubber- M_2	584g
Internal radius of the ball - R_1	11.1766cm
External radius of the ball - R_2	12.0160cm

Table 5 Basketball's characteristics

Having the necessary information, the theoretical numerical value for the ball's inertia moment was calculated to be $I_{pet} = 0.00869kgm^2$.

The experimental process consisted in placing the ball inside the inertial frame as shown in Fig 8 and measuring the oscillation period. The resulting period was$P = 3.56s$

Fig. 8 Basketball inside the torsional pendulum system

Making use of equation (18)

$$P = 2\pi\sqrt{\frac{(I_{pee} + I_F)}{C}}$$

The unknown variable I_{pee} was solved for, since the values for I_F, C and P were known. The experimental value obtained was $I_{pee} = 0.00803kgm^2$.

It is easy to see that the theoretical value (I_{pet}) and the experimental value (I_{pee}) are very similar, meaning that the torsion constant C and the inertial frame's moment I_F were calculated correctly. It is pretty clear that this system can be therefore used to obtain the inertia moments of any body, since it provides a close approximation to its real value. The system is now calibrated for its use on our aquatic vehicle.

3.3 Measurement of the mini submarine's inertia moment

The main drive for the creation of this system was to use it to properly and accurately measure the inertia moments of a mini submarine. Once the torsional pendulum was calibrated the measurement of the yaw moment of inertia. Following the same procedure as that of the basketball, and making use of equation (18). The obtained result was: $I_{sub} = 0.158kgm^2$

Fig. 9 Mini submarine inside the torsional pendulum

4. Results and Conclusion

In this work, a system was developed in order to measure the elements of the inertia matrix of a nonhomogeneous rigid-body, based on an idea suggested by D.A. Wells in the book “Lagrangian Dynamics” [1]. The torsional pendulum constructed is characterized most notably by two parameters: the inertia moment I_F and the torsion constant of the string C, which were determined by means of the known inertia moments of a homogeneous cylinder. For the calibration of the torsional pendulum an analytical expression was found for the inertia moment of a sphere composed of two layers (basketball), with said expression the theoretical calculation was possible and making use of the torsional pendulum the experimental value was obtained, finding both results to be similar. Once calibrated, the yaw inertia moment of the mini submarine was measured. The developed torsional pendulum showcased in this work is based off of an expression, which is valid for small torsional angles, and the measured oscillation period. The prototype presents a deficiency in its capacity to hold the nonhomogeneous body, nonetheless, this prototype is easy to build and can be improved by means of electronic devices.

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