Inter*net* Electronic Journal*

Nanociencia et Moletrónica

Diciembre 2012, Vol.10, N°2, pp 1915-1926

Nonlinear soliton tunneling effects: Applications to temporal and spatial optical solitons and matter-wave solitons in Bose-Einstein condensates

T. L. Belyaeva,[1] C. Hernandez-Tenorio,[2] L. M. Kovachev,[3] R. Perez-Torres,[4]
and V. N. Serkin[5]

[1]Universidad Autónoma del Estado de Mexico, C. P. 50000 Toluca, **México**
[2]Instituto Tecnológico de Toluca, Metepec, Edo. México, C. P. 52140, México
[3]Institute of Electronics, Bulgarian Academy of Sciences,
boul. Tzarigrasko Chaussee 72, Sofia 1784, **Bulgaria**
[4] Universidad Tecnológico del Valle de Toluca, Lerma, Edo. México, México
[5]Benemerita Universidad Autónoma de Puebla, C.P. 72502, 72001, Puebla, México

recibido: 10.11.12 revisado: 29.11.12 publicado: 31.12.12

Citation of the article;
T. L. Belyaeva, C. Hernandez-Tenorio, L. M. Kovachev,R. Perez-Torres,
and V. N. Serkin, Nonlinear soliton tunneling effects: Applications to temporal and spatial optical solitons and matter-wave solitons in Bose-Einstein condensates Int. Electron J. Nanoc. Moletrón, 2012, Vol. 10, N°2, pp 1915-1926

Nonlinear soliton tunneling effects: Applications to temporal and spatial optical solitons and matter-wave solitons in Bose-Einstein condensates

T. L. Belyaeva,[1] C. Hernandez-Tenorio,[2] L. M. Kovachev,[3] R. Perez-Torres,[4]
and V. N. Serkin[5]

[1]Universidad Autónoma del Estado de Mexico, C. P. 50000 Toluca, **México**
[2]Instituto Tecnológico de Toluca, Metepec, Edo. México, C. P. 52140, México
[3]Institute of Electronics, Bulgarian Academy of Sciences,
boul. Tzarigrasko Chaussee 72, Sofia 1784, **Bulgaria**
[4] Universidad Tecnológico del Valle de Toluca, Lerma, Edo. México, México
[5]Benemerita Universidad Autónoma de Puebla, C.P. 72502, 72001, Puebla, México

recibido: 10.11.12 revisado: 29.11.12 publicado: 31.12.12

Internet Electron. J. Nanoc. Moletrón., 2012, Vol.10 , N° 2 pp 1915-1926

ABSTRACT

Recent studies have thrown doubt on the ideal treatment of soliton tunneling. The most important enigmas in this field can be formulated in the following way: As to whether nonlinear soliton tunneling effect will resemble more the point like classical particle case or the quantum mechanical behavior in which the particle itself has an internal structure? How “hidden” degrees of freedom can show up in the process of soliton tunneling? What happens if the amplitude and duration of the input soliton vary in time when the soliton approaches a classically forbidden barrier? In particular, what happens in the case of the nonlinear tunneling of self-compressing soliton when its binding energy is increased? As to whether this case will resemble more the classical particle case or the quantum mechanical behavior? These questions are taken up in this Report.

Keywords: soliton tunneling effects, particle and wave behavior of solitons, nonlinear Schrödinger equations.

T. L. Belyaeva, C. Hernandez-Tenorio, L. M. Kovachev,R. Perez-Torres,
and V. N. Serkin
Internet Electron. J. Nanoc. Moletrón. 2012, Vol. 10, N° 2, pp 1915-1926

1. INTRODUCTION

Quantum mechanical tunneling has been one of the most fruitful concepts in modern physics. This discovery came at a time when the new quantum mechanics had been enjoying only its first steps in explaining the atomic structure. In the years since its discovery by Condon, Gamow, and Gurney[1], the concept of quantum tunneling has proven to be of great significance in many branches of physics, both from a fundamental point of view and from the point of view of applications, such as scanning tunneling microscope. While the study of linear microscopic quantum tunneling has a very long history going back to the first days of quantum mechanics, the field of nonlinear (soliton-like) macroscopic quantum tunneling is relatively young and, as a matter of fact, was born only after two discoveries - solitons and BEC.

The discovery of Bose-Einstein condensates (BEC) in trapped clouds of ultracold alkali atoms[2] opened unique possibilities to test the wave nature of matter and investigate quantum phenomena at macroscopic level.

The classical soliton concept was introduced by Zabusky and Kruskal[3] to characterize nonlinear solitary waves that do ot disperse and preserve their identity during propagation and after a collision. The optical soliton in telecommunication fibers presents a beautiful example in which an abstract mathematical concept has produced a large impact on the real world of high technologies[4-6]. It should be emphasized that the complementarity of these three fundamental discoveries - tunneling, solitons and BEC - will play the key role in the development of future technological applications of BEC where many ideas benefit each other.

Solitons behave as classical particles unless their phase starts to play a role. The effect of the phase appears more prominently in soliton-soliton interactions and in the interaction of solitons with a potential. The inhomogeneity provided by the trapping potential further enriched the solitons dynamics, soliton-soliton interaction, and solitons interaction with potentials. For instance, it is established that the phase difference between neighboring solitons of an attractive Bose-Einstein condensate is responsible for the observed repulsion between the solitons[7,8]. Phase plays an important role in the dynamics of solitons in different nonautonomous physical systems with varying dispersion, nonlinearity and gain or absorption[9]. The formation of matter-wave solitons by magnetically tuning the interatomic interaction near a Feshbach resonance provides a good example of a nonautonomous system[7,8].

Scientific terms are defined as they first appear. The term nonlinear soliton tunneling (NST) was introduced for the first time by Newell[10]. The propagation of a soliton toward a finite potential barrier (depending only on the spatial coordinate) was considered in this paper in the framework of the nonlinear Schrödinger equation model (NLSE) with linear and parabolic barriers. It was found that in certain circumstances, depending on the ratio of soliton amplitude to barrier height, the soliton can tunnel through the barrier in a lossless manner.

Since the first experimental discovery of macroscopic quantum tunneling (MQT) in Nb Josephson junctions by Voss and Webb[11], this phenomenon has been usually associated with the tunneling between different states of the system and similar effects have been discovered in completely different physical systems, such as liquid helium and nanomagnets[12]. BECs exhibit different kinds of tunneling phenomena[13-29]. It was shown that macroscopic quantum tunneling of Bose-Einstein condensates is the tunneling of a many-body wave function through a potential barrier, and therefore this effect provides a more stringent test of the validity of quantum mechanics than the one particle case[13,14]. This effect is referred to as collective quantum tunneling (CQT)[15]. This is to be contrasted with tunneling of the whole condensate in a variational parameter space, as was previously considered in the context of the collapse of a metastable attractive BEC in three dimensions[17].

In the pioneering experiment[18], Anderson and Kasevich investigated the dynamics of a BEC, which was accelerated by gravity in the periodic potential formed by two contra propagating vertical laser beams. In this way, Bloch oscillations of the condensate were induced, and each time a turning point of the oscillation was reached a fraction of the atoms tunneled into a continuum Bloch band. The regular output of atom pulses spectacularly proved the macroscopic coherence of the initially prepared condensate. In the experiment[18] the influence of atomic interactions mainly showed up as degradation in the interference when condensates with high densities were studied.

Direct observation of tunneling and nonlinear self-trapping in a single bosonic Josephson junction, implemented by two weakly linked Bose-Einstein condensates in a double-well potential, was reported in [26]. The question of how tunneling is affected by the interatomic interaction was investigated in [19-25]. In particular, it was shown that the interatomic interaction tends to suppress tunneling of the condensate in a double-well potential[19-25]. This effect can be so strong that under certain conditions the condensate remains self-trapped in one of the minima of the potential. The only difference with respect to superconducting Josephson junctions is that, for large initial imbalance, the condensate is mainly trapped in one of the wells, producing what is called macroscopic self-trapping (MST) [19-25]. Resonant tunneling effects of Bose-Einstein condensates in optical lattices and, in particular, resonantly enhanced tunneling effects in periodic potentials have been investigated by Zemesini *et al.*[27].

Salasnich *et al.* proposed the specific system of the condensate falling under gravity and scattering on a Gaussian potential barrier in [28]. It was shown that macroscopic quantum tunneling of a cigar-shaped BEC confined by two Gaussian barriers with periodically changing height opens novel feasibility to generate periodic soliton-like waves in atomic laser. Matter-wave bright solitons are macroscopic quantum objects that may acts as classical particle-like objects maintaining their integrity during collisions or while subjected to external forces. Lee and Brand[29] predicted enhanced quantum reflection of bright matter-wave solitons from a purely attractive potential in well. Different schemes to control the dynamics of matter-wave lattice solitons and remarkable possibilities to create quantum switches and quantum memories for matter-wave lattice solitons (based on the soliton tunneling and trapping effects) were first presented by Anufinger et al.[16]. The quantum nature of the matter-wave lattice solitons was explicitly manifested in the appearance of overbarrier reflection and tunnelling in this paper.

Dekel and coauthors[14] investigated nonlinear dynamic phenomena in macroscopic tunneling and demonstrated that the similarity between two physical objects, optical solitons and matter-wave solitons, both described by the mathematically similar equations, has made it possible to study both systems in parallel and shed light on one system by performing experiments on the other. An analytical study of macroscopic tunneling dynamics within the hydrodynamic representation has been presented also in [14].

Due to the evident complexity of experiments with matter wave solitons, Demidov *et al*.[30] demonstrated recently that spin waves are especially attractive for experimental studying of the effect of soliton tunneling through a potential barrier or well. As a striking example, Demidov et al.[30] discovered experimentally that the interaction of magnetic solitons with external potentials differs significantly from that of linear wave packets. In particular, the solitons demonstrate an enhanced tunneling through a potential barrier and an enhanced reflection from a potential well.

Another example is the recent paper by Barak *et al.* [31], where nonlinear soliton tunneling effects and optical soliton ejections were discovered experimentally. In this experiment, a wave packet was launched inside a trap potential. By increasing the power of the wave packet, the dynamics transformed from linear tunneling to nonlinear tunneling, and then to soliton ejection. In addition, partial trapping of the soliton was also experimentally observed. It seems very attractive to use the nonlinear optical soliton tunneling effect in developing basically novel all-optical tera-bit-rate soliton logic and switching devices[32]. Particle and wave properties of the NLSE solitons have been studied in the framework of different nonlinear models (see, for example, the reviews[33-35] and references therein).

The question we should like to discuss in the present paper is, what happens if the potential barrier is not only a function of coordinate, but it is also a function of time, and that is more, it is also a function of amplitude and duration of the tunneling soliton itself? And what happens if the amplitude and duration of the input soliton vary in time when the soliton approaches a classically forbidden barrier? In particular, what happens in the case of the nonlinear tunneling of self-compressing soliton? We carry out numerical simulations of the 1D nonautonomous Gross-Pitaevskii equation with varying in time nonlinearity and external potential barrier and reveal all possible macroscopic tunneling scenarios of matter waves including linear tunneling, nonlinear tunneling, soliton trapping, and soliton ejection. The growing in time nonlinearity provides the soliton self-compression during its tunneling. In experiments with Bose-Einstein matter waves, this can be achieved by using the Feshbach resonances to modulate the scattering length with time.

2. SOLITON TUNNELING EFFECT

The standard textbook approach describes the tunneling process for a pointlike particle in an external static potential. The canonical presentation of quantum tunneling is defined as a wave function passing through a classically forbidden potential energy barrier, in other words, when a maximum of the potential barrier is larger than the the energy of point-like particle, the Schrödinger equation allows nonvanishing wave functions in classically forbidden region on the other side of the barrier, that is why the quantum mechanical transmission probability need not to be zero[36]. If, on the other hand, the energy of a point-like particle is larger than the maximum of potential barrier (or well), then there is no turning point and reflection is forbidden in classical mechanics. However, the quantum mechanics allows the reflection of a particle from a classically allowed region even when there is no classical turning point. Quantum reflection can occur above a repulsive potential barrier or an attractive potential well.

It should be emphasized that quantum tunneling effect is a subject of constantly renewed interest, both experimentally and theoretically (see details of modern Rutherford experiment in[37] and the references therein). The most important problems are connected with quantum tunneling and reflection of a composite particle, in which the particle itself has an internal structure. Recently, Bertulani and coauthors[38] predicted important possibilities of enhancement of the tunneling due to the presence of intrinsic degrees of freedom.

In addition to chemical and nuclear subbarrier reactions, studies of the tunneling of extended and composite objects are of fundamental interest to astrophysical conditions[38].

Notice, that relative importance of the soliton concept in describing of extended elementary particles has long been furiously debated. The soliton does not behave like a classical pointlike particle; the soliton, as an extended and composite nonlinear object, possesses a wave-mechanical behavior as well.

Historically, the first demonstration that the soliton is a composite particle "possessing an inner degree of freedom" was found by Kosevich[39]. Kosevich considered the resonant and non-resonant soliton scattering by impurities in the mean field approximation and concluded that "we have to interpret the soliton as a bound state of N quasiparticles with its amplitude being proportional to N, that is why, when "the binding energy of a quasiparticle in the soliton is considerably larger that its kinetic energy it is difficult "to tear out" a free quasi-particle from the soliton"[39].This physical mechanism of the soliton decay during its scattering by external potentials is hereafter referred to as Kosevich mechanism of the soliton decay. Perhaps, Kosevich was the first to formulate the nonlinear Ehrenfest theorem. It was shown that if the potential does not depend on time, the law of the soliton center can be considered as the generalization of the Ehrenfest theorem for nonlinear system described by the mean field approximation (see Eqs.(14) and (19) in the paper[39] for details). To stress the complexity of the problem Kosevich concluded that "the NLSE soliton is similar to an atom but not to an elementary particle"[39].

It should be particularly emphasized that the soliton binding energy effect is most pronounced in nonlinear pairing of bright and dark NLSE solitons[40].

3. SOLITON NONLINEAR TUNNELING THROUGH EXTERNAL POTENTIALS: THE ROLE OF THE SOLITON BINDING ENERGY

Below we consider novel soliton tunneling scenarios, which illustrate how `hidden' degrees of freedom can show up in the process of nonlinear soliton tunneling. The canonical nonlinear Schrödinger equation

$$i\frac{\partial\psi(\xi,\eta)}{\partial\eta}=-\frac{1}{2}\frac{\partial^2\psi(\xi,\eta)}{\partial\xi^2}-|\psi(\xi,\eta)|^2\psi(\xi,\eta)+U_{ext}(\xi) \tag{1}$$

with $U_{ext}(\xi)=0$ had its beginning as an analytical description of optical and BEC solitons for autonomous systems. Without the self-interaction potential $|\psi(\xi,\eta)|^2$, Eq. (1) is similar to a quantum mechanical Schrödinger equation written in dimensionless form. This fact was used by Zakharov and Shabat when they proposed to interpret Eq.(1) as the nonlinear Schrödinger equation[41]. Let us calculate the mean energy as the expectation value of the Hamiltonian operator

$$\langle E\rangle=\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle}=\frac{\int_{-\infty}^{\infty}\left[|\psi_\xi(\xi,\eta)|^2-|\psi(\xi,\eta)|^4\right]d\xi}{2\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^2 d\xi} \tag{2}$$

The first term in brackets of Eq. (2) determines the mean kinetic energy, the second term can be considered as the binding energy of the quasi-particles associated with the soliton.

As is well known, Eq. (1) with $U_{ext}(\xi)=0$ possesses the so-called fundamental soliton solution

$$\psi_{sol}(\xi,\eta)=\kappa_0\,\mathrm{sech}(\kappa_0\,\xi)\exp[i(k\xi-\omega\eta)], \tag{3}$$

where κ_0 is the soliton form-factor, which determines the soliton amplitude distribution and is related with the soliton width as $\xi_{sol}=1/\kappa_0$, and ω is the soliton frequency: $\omega=(k^2-\kappa_0^2)/2$, and k is the soliton wave number. The soliton velocity is defined as $v_0=\partial\omega/\partial k=k$, and $\xi=(\xi-\xi_0-v_0\eta)$, where the initial center of mass position is given by ξ_0.

The canonical NLSE without external potential $U_{ext}(\xi)=0$, is integrable owing the fact that it has an infinite number of conserved (so-called polynomial) integrals. The three lowest conserved quantities are

$$\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^2 d\xi = 2\kappa_0 = N, \tag{4}$$

$$\langle P\rangle = \frac{i}{2}\frac{\int_{-\infty}^{\infty}\psi(\xi,\eta)\psi^*_{\xi}(\xi,\eta)-\psi_{\xi}(\xi,\eta)\psi^*(\xi,\eta)d\xi}{\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^2 d\xi} = v_0, \tag{5}$$

$$\langle E\rangle = \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \langle\hat{T}_{kin}\rangle + \langle\hat{E}_b\rangle = \frac{1}{2}\frac{\int_{-\infty}^{\infty}|\psi_{\xi}(\xi,\eta)|^2 d\xi}{\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^2 d\xi} - \frac{1}{2}\frac{\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^4 d\xi}{\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^2 d\xi}, \tag{6}$$

where N in Eq. (4) is a constant, which can be interpreted as the total number of atoms in condensate or the total number of photons in laser pulse.

In the total soliton mean energy Eq. (6), we specially separated two terms connected with linear and nonlinear features of the problem. It should be emphasized, that in contrast to the ordinary quantum mechanical condition for the probability density $\int_{-\infty}^{\infty}|\Psi(\xi,\eta)|^2 d\xi = 1$, the condition Eq.(4) should be applied in Eq.(6). Then we obtain for the kinetic and binding energy of the soliton, correspondingly,

$$\langle\hat{T}_{kin}\rangle = \frac{1}{2}\frac{\int_{-\infty}^{\infty}|\psi_{\xi}(\xi,\eta)|^2 d\xi}{\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^2 d\xi} = \frac{1}{6}\kappa_0^{\ 2} + \frac{1}{2}v_0^{\ 2} \tag{7}$$

$$\langle\hat{E}_b\rangle = -\frac{1}{2}\frac{\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^4 d\xi}{\int_{-\infty}^{\infty}|\psi(\xi,\eta)|^2 d\xi} = -\frac{1}{3}\kappa_0^{\ 2}. \tag{8}$$

We have to remember that the classical and quantum mechanical mean values are interpreted differently, that is why the mean soliton kinetic energy is not equal to classical one: $\langle T_{kin}\rangle \neq v_0^2/2$. What's more, if we set $v_0 = 0$, we obtain $\langle T_{kin}\rangle = \kappa_0^{\ 2}/6 > 0$. The mean kinetic energy does not vanish for the soliton being at rest in absolute analogy with a quantum mechanical particle being in rest. Physically, this feature is connected with the Heisenberg uncertainty principle that is why Eq. (7) can be considered as the generalization of this general principle on to the NLSE soliton dynamics.

We stress, that to date, we can find many misleading interpretations of binding soliton energy, that "roam from place to place" and these misinterpretations still present the fundamental enigmas of solitons.

In low temperature physics, for the first time Eq.(1) was named by Tsuzuki[42] as the Pitaevskii-Gross equation for the condensate at absolute zero temperature (where $U_{ext}(\xi)$ = μ is the chemical potential assumed to be constant). Historically, Zakharov and Shabat proposed the name "nonlinear Schrödinger equation" for Eq.(1) at the limit $U_{ext}(\xi) = 0$ to stress its analogy with ordinary quantum mechanical Schrödinger equation with the equivalent self-trapping potential $U(\xi,\eta)=|\psi(\xi,\eta)|^2$. Considering the conservation laws for the NLSE, Zakharov and Shabat proposed to interpret the first three integrals "apart from coefficients"[41] as the number of particles, the momentum and the energy. After this work, in all subsequent works, the momentum and the energy of the soliton have been directly identified with two integrals of motion of the NLSE:

$$P = \frac{i}{2}\int_{-\infty}^{\infty}\left[\psi(\xi,\eta)\psi^*_{\xi}(\xi,\eta)-\psi_{\xi}(\xi,\eta)\psi^*(\xi,\eta)\right]d\xi = C_2, \tag{9}$$

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\left[|\psi_{\xi}(\xi,\eta)|^2 - |\psi(\xi,\eta)|^4\right]d\xi = C_3 \tag{10}$$

with the ordinary quantum mechanical condition for the probability density $\int_{-\infty}^{\infty}|\Psi(\xi,\eta)|^2 d\xi = C_1 = 1.$

But we stress that in the case of the nonlinear NLSE, the main normalization condition is not $C_1 \neq 1$ as in ordinary quantum mechanics and, obviously, depends on the soliton form-factor $C_1 = \int_{-\infty}^{\infty}|\psi(\xi,\eta)|^2 d\xi = 2\kappa_0 = N.$ That is why the soliton momentum, kinetic energy and its binding energy are given by (4-8) and both interpretations of the soliton form factor as its mass $M = 2\kappa_0$ (misinterpretation), the momentum as $\langle P\rangle = Mv_0 = 2\kappa_0 v_0$ (mis-interpretation), the kinetic energy as $\langle T_{kin}\rangle = Mv_0^2/2 = \kappa_0 v_0^2$ (misinterpretation), and the soliton binding energy as $\langle E_b\rangle = -\frac{1}{2}\int_{-\infty}^{\infty}|\psi(\xi,\mu)|^4 d\xi = -2\kappa_0^3/3$ (misinterpretation) can be considered as long living enigmas of the Schrödinger solitons.

Similar to the nuclear binding energy, the soliton binding energy $\langle E_b\rangle = -\kappa_0^2/3$ (8) denotes the degree of how strongly the quasi-particles bind together. Just as the increasing of the soliton form-factor κ_0 results in a growth of the soliton binding energy, so also the enhancement of nonlinearity results in the same effect. In optical applications, the soliton binding energy increases with peak power growing and with decreasing of the group velocity dispersion along the propagation distance[43,44]. In BEC's applications, the matter wave soliton binding energy increases both under the Feshbach resonance management and with growing of atomic density.

In classical mechanics, if we consider a moving particle encountering a potential barrier or a potential well, the barrier slows down the particle and if the energy of the particle is too low relative to the barrier's height, the particle is reflected. In the classical scenario, in the potential well case, the particle speeds up in the well and is always transmitted. In the quantum mechanical tunneling scenario, in both cases we have reflection and transmission behavior of the wave function $\psi(\xi,\eta)$ dependent on the parameters of the potential and on the energy of the incident particle.

In nonlinear optical and BEC applications, the NLSE (1) can be written down in the most general form taking into account the varying dispersion $D(\eta)$, nonlinearity $R(\eta)$, gain or loss $\alpha(\eta)$, and trapping potential $U(\xi)$

$$i\frac{\partial\psi(\xi,\eta)}{\partial\eta} = \left[-\frac{1}{2}D(\eta)\frac{\partial^2}{\partial\xi^2} - G(\eta)|\psi(\xi,\eta)|^2 + U_{ext}(\xi) + \frac{i}{2}\alpha(\eta)\right]\psi(\xi,\eta) \quad (11)$$

In our numerical experiments, the soliton is initialized far from the localized barrier and is allowed to propagate toward the scattering potentials localized in the region $|\xi| \le 1$ with different height U_0 and strengths ε: $U(\xi)=U_0(1-\varepsilon^2\xi^2)$ under condition that $U(\xi) = 0$ in the intervals $|\xi| > 1$. For the purposes of our analysis, it is convenient to use in the case of constant gain α_0 and $R(\eta) = G(\eta)\exp\int_0^{\eta}\alpha(\tau)d\tau$ in Eq.(11).

In Figs.1, 2 we illustrate tunneling of the self-compressing due to increasing nonlinearity $R(\eta)=\exp(\alpha_0\eta)$ solitons:

1. At very low nonlinearities $R \to 0$, the soliton-like (*sech*-like) initial wave packet shows ordinary dispersion spreading behavior and behaves as a slowly modulated plane wave which splits into a reflected and a transmitted parts as shown in Fig.1a and Fig.2a. At $G = 1$ and $\alpha_0 \simeq 0$, $R\to 1$, without self-compressing effect, all solitons with kinetic energy $E_{kin} \le U_0$ are split into transmitted and reflected solitons with small intensities (Fig. 1b and Fig.2b).

2. With larger parameters α_0 and increasing nonlinearity $R(\eta)=\exp(\alpha_0 \eta)$, on the other hand, the soliton is almost completely transmitted through classically forbidden barrier (see Fig.1c and 2c). This tunneling scenario shows the possibility of the soliton sub-barrier transmission at the intermediate region of the nonlinearity strength $R(\eta)=\exp(\alpha_0 \eta)$. The trajectory of the soliton in this sub-barrier transmission regime is extremely sensitive to the strength of the nonlinearity given by parameter α_0.

3. For large parameters α_0, there is a critical strength of varying nonlinearity above which all self-compressing solitons are reflected and never transmitted (Fig. 1(d) and Fig. 2(d)). We have found extremely elastic soliton reflection behavior in all the cases of higher parameters α_0. Essentially no energy is lost during the soliton reflection; the whole kinetic energy of the incoming soliton is converted into the reflected one and, because of this, the absolute value of its velocity is unchanged (Fig. 1d and Fig. 2d). All three cases (1-3) considered, we obtain equal dimensionless "masses" $m = 1$ given by Eqs.(1,11) and hence equal "classical" kinetic energies $v_0^2/2 = 0.5$.

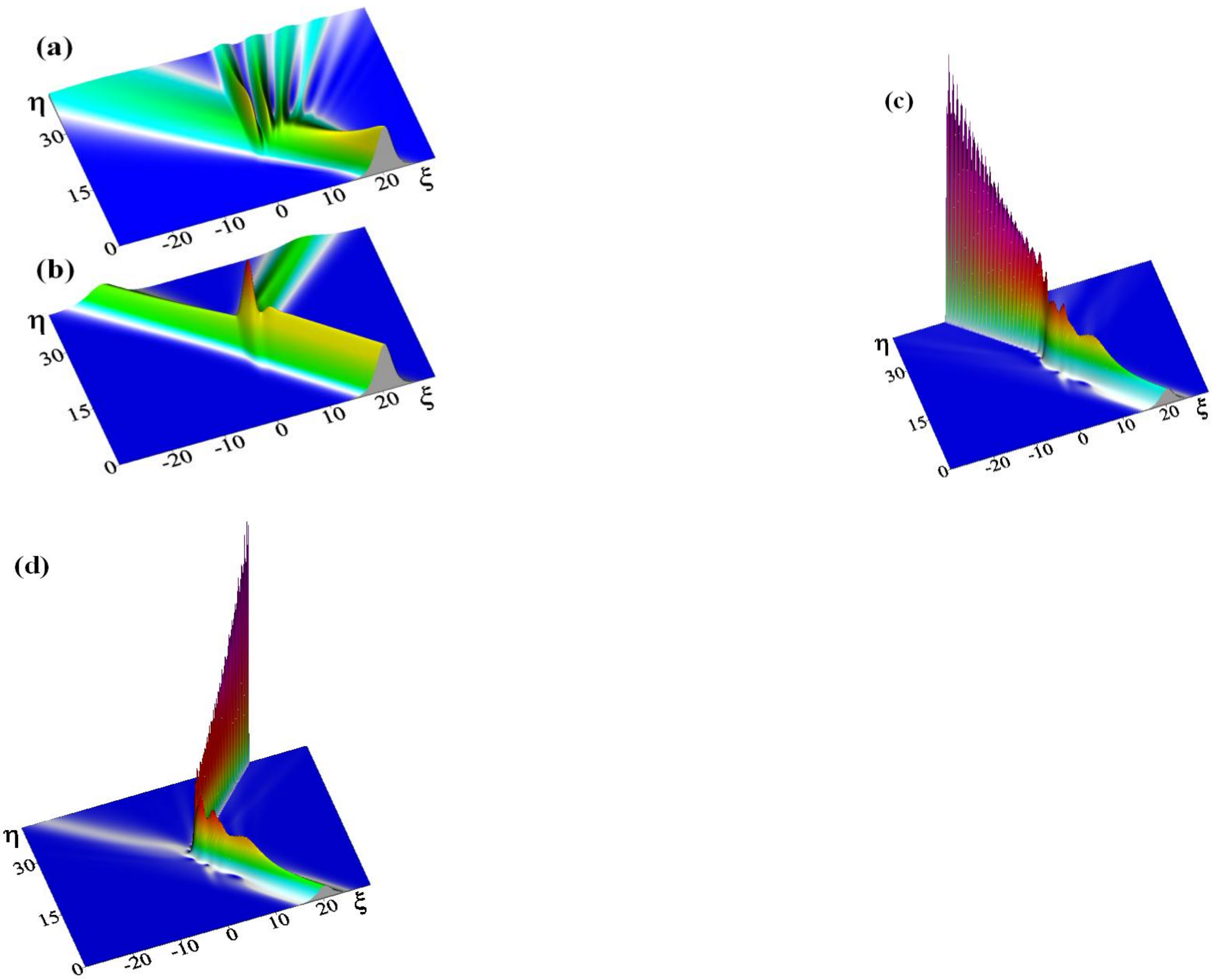

Fig.1. Dependence of the main features of nonlinear soliton tunneling effect on the soliton binding energy calculated in the framework of the model Eq. (12): (a) linear wave packet tunneling scenario $R = 0$, $\alpha_0 = 0.0$; (b) soliton decay into transmitted and reflected solitons, $\alpha_0 = 0.0$; (c) soliton self-compression and sub-barrier tunneling effect, $\alpha_0 = 0.075$; (d) soliton self-compression and complete particle like reflection scenario, $\alpha_0 = 0.08$. Parabolic potential barrier $U(\xi)=U0(1-\xi^2)$ is localized in the region $-1 \leq \xi \leq 1$ with height $U_0 = 0.6$.

To have an analogy with quantum over-barrier reflection effect, let us consider the case where the incoming soliton has kinetic energy, which is greater than the barrier height. In the kinetic energy region $E_{kin} \geq U_0$, (for example, in Fig.3: $E_{kin} = v_0^2/2 = 0.605$ at $v_0 = 1.1$; and $U_0 = 0.6$, (Fig.3)) the simulations show a more complicated nonlinear scattering process. In this case, the soliton behavior is very different from the classical mechanical one: the compressing in time soliton splits into reflected and transmitted (also self-compressing) solitons (Fig. 3(a,b)). Hence, the effect of the soliton over-barrier reflection can be observed in this case (Fig. 3b). When the parameter α_0 is increased the soliton tunneling behavior is changed dramatically: we have found that the intensity of the reflected soliton is also increased with time; this fact is represented by yellow color in Fig 3b showing the counter plots of the $|\psi(\xi,\eta)|^2$. Notice, that in this case the transmitted soliton exhibits oscillations with very regular amplitudes and periods (Fig. 3(b)). These oscillations can be explained by the well-known effect connected with soliton amplitude asymptotic transformations induced by its form factor perturbations[4-6]. The behavior changes when α_0 is increased more and more. For large parameters α_0 the soliton appears to be completely transmitted and never reflected (see Fig. 3c). In other words, a self- compressing soliton with high amplitude resembles a classical particle in the sense that the soliton maintains its integrity and follows a well-defined classical trajectory. In this case, the self-compressing soliton remains mainly classical particle like, with very little loss of energy to dispersive waves (see Fig.3c). There exists a critical value α_0, above which the soliton is transmitted classically. The larger is the soliton's self-compression effect given by parameter α_0, the smaller is the reflected and transmitted nonsoliton radiation and, accordingly, the larger is the transmitted soliton intensity.

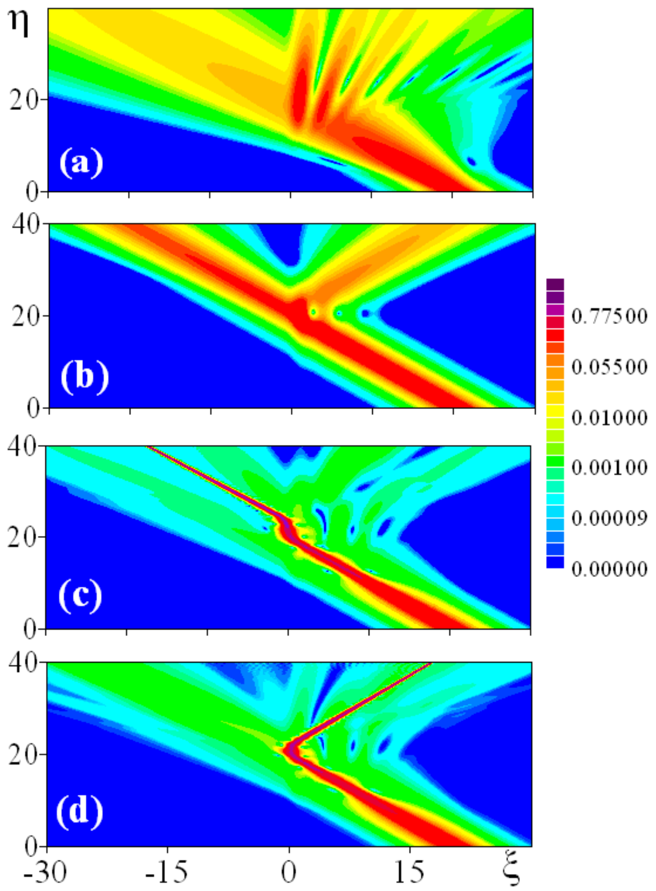

Fig.2 Comparison of contour maps for linear and nonlinear tunneling through parabolic localized potential barrier calculated in parameter regions given in Fig.1: (a) linear quantum mechanical tunneling of spreading wave packet, (b) soliton splitting on parabolic barrier; (c) soliton self-compression and sub-barrier tunneling effect calculated in parameter regions $\alpha_0 = 0.075$; (d) complete particle like reflection of self-compressed soliton calculated after choosing $\alpha_0 = 0.08$.

Summarizing our computer experiments, we have thus shown that the soliton tunneling effect has many interesting properties that need to be investigated further, for example, by applying direct analytical methods based on the IST method. Tunneling of optical solitons was recently observed experimentally and these different tunneling regimes were obtained by changing the intensity of the incident soliton[31].

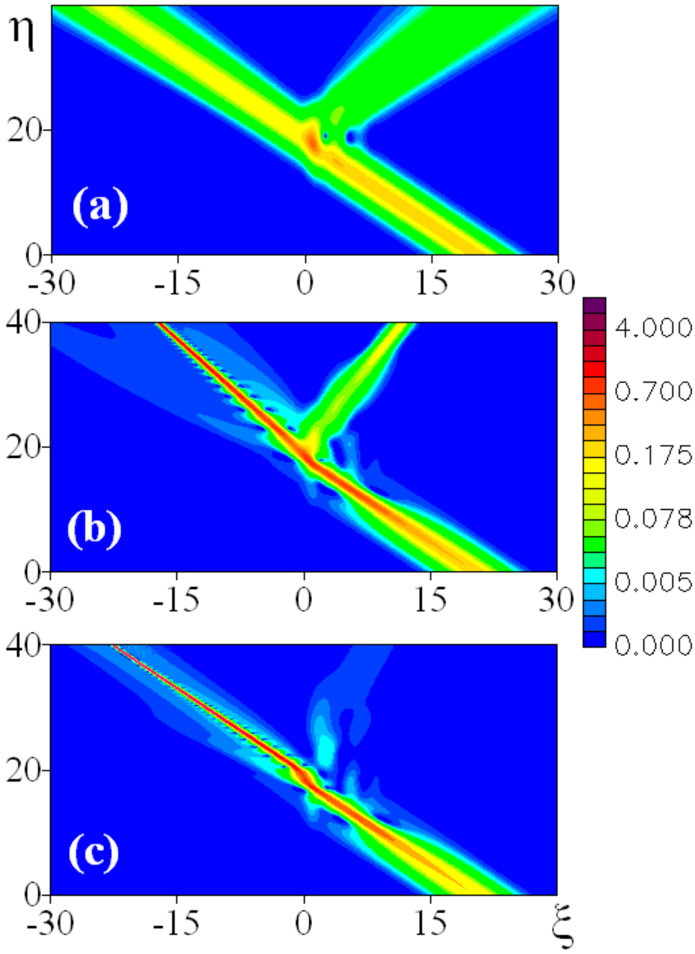

Fig.3 Soliton over-barrier reflection effect calculated in parameters region $U_0 = 0.6$; $\varkappa_0 = 0.5$; $v_0 = -1.1$, $v_0^2/2 = 0.605$ and (a) $\alpha_0 = 0.0$, (b) $\alpha_0 = 0.07$, and (c) $\alpha_0 = 0.08$.

4. CONCLUSION

In this paper, we have considered the nonlinear soliton tunneling effect through a potential barrier taking into account the soliton self-compression. We reveal, for the first time of our knowledge, previous misinterpretations of the binding soliton energy and investigate all possible scenarios of the nonlinear soliton tunneling effect. In particular, we have investigated all possible soliton tunneling scenarios including linear-like tunneling and soliton splitting, nonlinear tunneling, soliton trapping, and soliton ejection. Amplification of a Schrödinger soliton, which is mathematically equivalent a growing in time nonlinearity, provides the soliton self-compression during its transmission through or reflection from the parabolic barrier. In experiments with Bose-Einstein matter waves, this can be achieved by using the Feshbach resonances to modulate the scattering length with time. In nonlinear optics, soliton self-compression can be realized by increasing the power of the solitonic pulse or decreasing of the wave group velocity dispersion along the propagation distance[43,44]. We have found that, due to the soliton self-compression, nonlinear soliton behavior differs essentially from quantum mechanical linear tunneling, and in particular, we have found the possibility for solitons to propagate with sudden and full transmission corresponding to the observed soliton ejection[31].

REFERENCES

[1] Gamow, G., "Zur Quantentheorie des Atomkernes", Z. Physik 51, 204 (1928); Nature 122, 805 (1928); Gurney, R. W., Condon, E. U., "Quantum Mechanics and Radioactive Disintegration", Nature 122, 439 (1928); Phys. Rev. 33, 127 (1929).

[2] Cornell, E. A, Wieman, C. E., "Nobel Lecture: Bose-Einstein condensation in a dilute gas, the first 70 years and some recent experiments", Rev. Mod. Phys. 74, 875-893 (2002); Ketterle, W., "Nobel lecture: When atoms behave as waves: Bose-Einstein condensation and the atom laser", Rev. Mod. Phys. 74, 1131-1151 (2002).

[3] Zabusky, N.J. and Kruskal, M.D., "Interaction of "solitons" in a collisionless plasma and the recurrence of initial states", Phys. Rev. Lett. 15, 240-243 (1965).

[4] Hasegawa, A. [Optical Solitons in Fibers], Springer-Verlag, Berlin (1989); Hasegawa, A; Kodama, Y. [Solitons in Optical Communications], Oxford University Press, New York (1995); Hasegawa, A., Matsumoto, M. [Optical Solitons in Fibers], Springer-Verlag, Berlin (2003).

[5] Agrawal, G. P. [Nonlinear Fiber Optics], 3rd edition, Academic Press, San Diego (2001); Agrawal, G. P. [Applications of Nonlinear Fiber-Optics]; ibid. (2001).

[6] Akhmediev, N. N.; Ankiewicz, A. [Solitons. Nonlinear pulses and beams], Charman and Hall, London (1997).

[7] Strecker, K. E., Partridge, G. B., Truscott, A.G., and Hulet, R. G., “Formation and propagation of matter wave soliton trains”, Nature (London) 417, 150-151 (2002); "Bright matter wave solitons in Bose-Einstein condensates", New J. Phys. 5, 73.1-73.8 (2003).

[8] Khaykovich, L., Schreck F., Ferrari G., Bourdel T., Cubizolles J., Carr L. D., Castin Y., Salomon C. "Formation of matter wave bright solitons", Science 296, 1290-1293 (2002).

[9] Serkin, V. N. and Hasegawa, A., " Novel soliton solutions of the nonlinear Schrödinger ecuation model", Phys. Rev. Lett. 85, 4502-4505 (2000); "Exactly Integrable Nonlinear Schrödinger Equation Models with Varying Dispersion, Nonlinearity and Gain: Application for Soliton Dispersion Management", IEEE J. Selected Topics in Quantum Electron. 8, 418-478, (2002); Serkin, V. N. , Hasegawa, A., and Belyaeva, T. L.,"Nonautonomous Solitons in External Potentials", Physical Review Letters, 98, 074102 (2007).

[10] Newell, A. C., “Nonlinear Tunnelling”, J. Math. Phys., 19, 1126-1136 (1978).

[11] Voss, R. F., Webb, R. A., “Macroscopic Quantum Tunneling in 1-μm Nb Josephson Junctions”, Phys. Rev. Lett., 47, 265 (1981).

[12] Friedman, J. R., Han, S., [Exploring the quantum/classical frontier: recent advances in macroscopic quantum phenomena], Nova Science Publishers, Inc. (2002); Ankerhold, J., [Quantum tunneling in complex systems: the semiclassical approach], Springer, (2007); Tomsovic, S., [Tunneling in complex systems], World Scientific, 1998.

[13] Carr, L. D., Holland, M. J., Malomed, B. A., J. Phys. B 38, 3217 (2005); Moiseyev, N., Carr, L. D., Malomed, B. A., Band, Y. B. J. Phys. B: At. Mol. Opt. Phys. 37, L1 (2004); Lee, C. and J. Brand, Europhys. Lett. 73, 321 (2006).

[14] Dekel, G., Farberovich, O.V., Soffer, A., Fleurov, V., Physica D: Nonlinear Phenomena, 238, 1475 (2009); Fleurov, V., Soffer, A., Europhysics Letters 72, 287 (2005); Dekel, G., Fleurov, V., Soffer, A., Stucchio, C., Phys. Rev. A 75, 043617 (2007).

[15] Khomeriki, R., Ruffo, S., Wimberger, S., Europhysics Letters, 77, 40005 (2007).

[16] Ahufinger, V., Mebrahtu, A., Corbalán, R., Sanpera, A., New Journal of Physics 9, 4 (2007).

[17] Kagan, Y., Shlyapnikov, G. V., Walraven, J. T. M., Phys. Rev. Lett. 76, 2670 (1996); Shuryak, E. V., Phys. Rev. Lett. 54, 3151 (1996); Stoof, H. T. C., J. Stat. Phys. 87, 1353 (1997); Ueda, M., Leggett, A. J., Phys. Rev. Lett. 80, 1576 (1998); Sackett, C. A., Stoof, H. T. C., Hulet, R. G., Phys. Rev. Lett. 80, 2031 (1998).

[18] Anderson, B.P., Kasevich, M., Science 282, 1686 (1999).

[19] Smerzi, A., Fantoni, S., Giovannazzi, S., Shenoy, S. R., Phys. Rev. Lett. 79, 4950 (1997).

[20] Milburn, G. J., Corney, J. Wright E., Walls, D.F., Phys. Rev. A 55, 4318 (1997)

[21] Zapata, I., Sols, F., Leggett, A.J., Phys. Rev. A57, R28 (1998).

[22] Raghavan, S., Smerzi, A., Fantoni, S., Shenoy, S.R., Phys. Rev. A 59, 620 (1999).

[23] Salasnich, L., Parola, A., Reatto, L., Phys. Rev. A60, 4171 (1999).

[24] Cataliotti, F. S., Burger, S., Fort, C., Maddaloni, P., Trombettoni, A., Smerzi, A., Inguscio, M., Science 293, 843 (2001).

[25] Morsch, O., Mulle, J. H., Cristiani, M., Ciampini, D., Arimondo, E., Phys. Rev. Lett. 87, 140402 (2001).

[26] Albiez, M., Gati, R., Folling, J., Hunsmann, S., Cristiani, M., Oberthaler, M. K., Phy. Rev. Lett. 95, 010402 (2005).

[27] Zenesini, A. *et al.*, Phys. Rev. Lett. 103, 090403 (2009); A. Zenesini, *et al.*, New J. Phys. 10, 053038 (2008); C. Sias, *et al.*, Phys. Rev. Lett. 98, 120403 (2007).

[28] Salasnich L, Parola A, and Reatto L., Phys. Rev. A 64 023601 (2001); J. Phys. B: At. Mol.Opt. Phys. 35, 3205 (2002).

[29] Lee, C., Brand, J., Europhysics Letters 73, 321 (2006).

[30] Demidov, V. E., Hansen, U. H., Demokritov, S. O., Phys. Rev. B 78, 054410 (2008).

[31] Barak, A., Peleg, Or., Stucchio, C., Soffer, A., Segev, M., "Observation of soliton tunneling phenomena and soliton ejection", Phys. Rev. Lett., 2008, Vol. 100, 153901 (2008).

[32] Serkin, V. N., "Nonlinear tunneling of temporal and spatial optical solitons through organic thin films and polymeric waveguides", Proc. SPIE 3927, 292 (2000); H. Sakaguchi and M. Tamura, J. Phys. Soc. Jpn. 73, 503 (2004); ibid. 74, 292 (2005).

[33] Yang, R., Wu, X., Opt. Express 16, 17759 (2008).

[34] Kivshar, Y. S., Malomed, B. A., "Dynamics of solitons in nearly integrable systems", Rev. Mod. Phys. 61, 763 (1989).

[35] Goodman, R. H., Holmes, P. J., Weinstein, M. I., Physica D 192, 215 (2004); Stoychev, K.T., Primatarowa, M. T., Kamburova, R. S., Phys. Rev. E 70, 066622 (2004).

[36] S.B. McKagan, K.K.Perkins, and C.E. Wieman, Phys. Rev. ST Phys. Educ. Res. 4, 020103 (2008); L.D.Carr, S.B. McKagan, American Journal of Physics 77, 308 (2009).

[37] A. Lemasson, A. Shrivastava, M. Rejmund, N. Keeley, V. Zelevinsky, S. Bhattacharyya, A. Chatterjee, G. de France, B. Jacquot, V. Nanal, R. G. Pillay, R. Raabe, and C. Schmitt, Phys. Rev. Lett., 103, 232701 (2009).

[38] C. A. Bertulani, V. V. Flambaum and V. G. Zelevinsky, J. Phys. G: Nucl. Part. Phys., 34, 2289 (2007).

[39] Kosevich, "A. M., "Particle and wave properties of solitones", Physica D, Vol. 41, 253 (1990).

[40] Afanas'ev, V. V., Dianov, E. M., Prokhorov, A. M., Serkin, V. N., "Nonlinear pairing of light and dark optical solitons", JETP Lett., 48, 638 (1988); Afanasjev, V. V., Dianov, E. M., Serkin, V. N. "Nonlinear pairing of short bright and dark soliton pulses by phase cross modulation", IEEE J. Quant. Electron. 25 (12), 2656-2664 (1989); Afanasjev, V. V., Kivshar, Y. S., Konotop, V. V., Serkin, V., N., Opt. Lett., 14, 805 (1989); Busch, T., Anglin, J. R., Phys. Rev. Lett., 87, 010401 (2001).

[41] Zakharov, V. E., Shabat, A. B. Sov. Phys.-JETP 34, 62–69 (1972).

[42] Tsuzuki, T., J. Low Temper. Phys., 4, 441-457 (1971).

[43] Reeves-Hall, P. C.; Taylor, J. R. Electron. Lett. 37, 417-419 (2001); Reeves-Hall, P. C.; Lewis, S. A. E.; Chernikov S. V.; and Taylor, J. R. ibid. 36, 622-624 (2000); Travers, J. C.; Stone, J. M., Rulkov, A. B.; Cumberland, B. A.; George, A. K.; Popov, S. V.; Knight, J. C.; and Taylor, J. R. Opt. Express 15, 13203 (2007).

[44] Liu, X.; Xu, C.; Knox, W. H.; Chandaha, J. K.; Eggleton, B. J.; Kosinski, S. G.; Windeler, R. S. Opt. Lett. 26, 358—360 (2001); Sysoliatin, A.; Belanov, A.; Konyukhov, A.; Melnikov, L.; and Stasyuk, V.; IEEE J. Sel. Top. Quantum Electron. 14, 733—738 (2008); Sysoliatin, A. A.; Senatorov, A. K.; Konyukhov, A. I.; Melnikov, L. A.; and Stasyuk, V. A. Opt. Express 15, 16302—16307 (2007).