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ABSTRACT

Mathematical similarities and parallels between two different physical objects, optical solitons and matter-wave solitons, both described by mathematically similar models: the nonlinear Schrödinger equation (NLSE) and the Gross-Pitaevskii equation (GPE) model, open the possibility to study both systems in parallel and because of the obvious complexity of experiments with matter-wave solitons, offer outstanding possibilities in studies of BEC system by performing experiments in the nonlinear optical system and vise versa. In our paper, we review recent ideas to overcome the self-focusing collapse and generate stable soliton bullets adapted to external potentials. Based on the generalized nonlinear Schrödinger equation model with sign-reversal varying-in-time harmonic oscillator potential, we show that conditions of its exact integrability in one-dimensional case (1D) indicate conclusively the way for soliton-like bullets generation in 3D nonautonomous nonlinear and dispersive systems. It turns out that generation of matter-wave soliton bullets can be realized if periodic variations of nonlinearity and confining potential are opposite in phases so that peaks of nonlinearity inside the atomic cloud coincide in time with repulsive character of trapping potential.

1. INTRODUCTION

Mathematical analogies and the parallels between the Gross–Pitaevskii equation (GPE) governing the nonlinear excitations in Bose–Einstein condensates (BECs) and the famous nonlinear Schrödinger equation (NLSE) came to be regarded as the cornerstone of BEC physics and has brought together physicists from different areas, in particular, from atomic, nonlinear optical, laser and condensed matter physics, fluid mechanics and fundamental general and particle physics.

Three dimensional (3D) spatiotemporal solitons that are self-localized in two transverse dimensions and one longitudinal dimension were predicted by Silberberg in 1990 and termed *light bullets.* Spatiotemporal soliton bullets are among the most intriguing and challenging entities. There has been an extensive field of research in the past 10 years in the areas of light bullets and the limitations of the standard NLSE. In general, soliton bullets may be supported by a variety of nonlinear mechanisms. These include quadratic nonlinear media that support solitons for all physical dimensions and where two dimensional bullet formation was achieved, saturable and nonlocal media; materials with competing nonlinearities where higher-order effects such as fourth-order dispersion may play a stabilizing role, propagation in optical lattices and filamentation, the concept of three-dimensional light bullet formation in tandem structures where nonlinearity and dispersion are contributed by different materials, just to name a few. Experimentally, however, only 2D spatiotemporal solitons have been demonstrated. The interested reader can find the review of optical bullets principles and research as it currently stands in[1-23], and references therein.

The discovery of BEC in trapped clouds of ultracold alkali atoms opened unique possibilities to investigate the wave nature of matter[24-27]. This has been shown in the elegant BEC experiments that discover, among other things, dark and bright matter wave solitons[28-30]. To date, bright matter wave solitons and trains of bright solitons have been studied in experiments with Bosonic lithium having attractive interactions. The experimental realization was possible by tuning the scattering length with a Feshbach resonance, first producing a condensate with repulsive interactions, loading it into a one-dimensional geometry and then switching the interaction to attractive by a change in the magnetic offset field. Matter wave dark solitons have been realized in most experiments by using the so-called the phase imprinting method. However, much less attention has been paid to studying the behavior of solitons in BECs exhibiting both varying in time nonlinearity and accordingly varying in time trapping potential. Up to now, no attempts were made in experiments to relate these two different processes and to find nontrivial laws of soliton adaptation in external potentials. A subtle interplay between nonlinear management by means of the Feshbach resonance on the one hand, and linear management by means of confining time-dependent potential on the other hand, can result in a rich variety of matter wave solitons with several interesting properties. To test the validity of the theory, the experimental arrangement should be inspected to be as close as possible to the optimal map of parameters at which the problem proves to be exactly integrable. We propose the experimentally accessible methods to generate the nonautonomous matter wave solitons in BEC and demonstrate that variations of magnetically tuned scattering length must be consistent with variations of the confining potential. It is found, in particular, that near a Feshbach resonance the matter wave soliton can be stabilized even without an axial trapping potential. In the case of periodically varying nonlinearity, variations of the external harmonic potential are found to be periodically sign-reversal (expulsive/attractive) in time. The GP wave equation, as is well known, predicts the catastrophic collapse of self-focusing BEC. The loss of validity of the theory is studied in details and direct numerical experiments are presented which demonstrate the advantages and limitations of our technique.

Based on the generalized nonlinear Schrödinger equation model with sign-reversal varying-in-time harmonic oscillator potential, we show that conditions of its exact integrability in one-dimensional case (1D) indicate conclusively the way for soliton-like bullets generation in 3D nonautonomous nonlinear and dispersive systems during reversal periodic transformation from cigar-shaped to ball-shaped trapping potential. It turns out that the generation of matter-wave soliton bullets can be realized if periodic variations of nonlinearity and confining potential are opposite in phases so that the peaks of nonlinearity inside the atomic cloud coincide in time with repulsive character of trapping potential. In nonlinear optical applications, this kind of periodic graded-index nonlinear structure with alternating wave-guiding and anti-wave-guiding segments can be used to simulate different and complicated processes in the total scenario of matter-wave soliton bullets generation.

2. THE LAW OF SOLITON ADAPTATION TO EXTERNAL POTENTIALS

The existence of quasi-one-dimensional (1D) solitons emphasizes the fundamental feature of the 1D NLSE model known as its complete integrability. That was the main reason why Hasegawa and Tappert[31] proposed a cylindrically symmetric optical fiber to start theoretical and experimental studies with optical solitons[31-35], and that is why a quasi-one-dimensional (cigar-shaped) atom cloud was used in all pioneering and recent experiments with matter wave solitons (see [28-30] and references therein).

The classical soliton concept was developed for nonlinear and dispersive systems that have been autonomous; namely, time (or space coordinate in the case of spatial and temporal optical solitons) has only played the role of the independent variable and has not appeared explicitly in the nonlinear evolution equation. Recently, it was established that solitary waves in nonautonomous nonlinear and dispersive systems can propagate in the form of so-called nonautonomous solitons or soliton-like similaritons (see[36-60] and references therein). Nonautonomous solitons hold their shape self-similarly, interact elastically and generally move with varying amplitudes, speeds and spectra adapted both to the external potentials and to the dispersion and nonlinearity variations. The law of soliton adaptation to external potentials was obtained by different methods, in particular, by Inverse Scattering Transform (IST) method, Penleve analysis and similarity transformations. Let us summarize the main results obtained in this rapidly developing field.

In the framework of the nonautonomous NLSE model with linear and harmonic oscillator potentials

$$i\frac{\partial\psi}{\partial\eta}=\left[-\frac{1}{2}D(\eta)\frac{\partial^2}{\partial\xi^2}-\sigma G(\eta)\left|\psi(\xi,\eta)\right|^2+2\alpha(\eta)\xi+\frac{1}{2}\Omega^2(\eta)\xi^2+i\frac{1}{2}\Gamma(\eta)\right]\psi, \tag{1}$$

$$\psi(\xi,\eta)=Q(\xi,\eta)\exp\left(\int_0^\eta\Gamma(\tau)d\tau/2\right)$$

$$R(\eta) = G(\eta) \exp \int_0^\eta \Gamma(\tau) d\tau$$

$$i \frac{\partial Q}{\partial \eta} + \frac{1}{2} D(\eta) \frac{\partial^2 Q}{\partial \xi^2} + \left[\sigma R(\eta) |Q|^2 - 2\alpha(\eta)\xi - \frac{1}{2}\Omega^2(\eta)\xi^2 \right] Q = 0 \quad (2)$$

nonautonomous solitons exist if and only if dispersion $D(\eta)$, nonlinearity $R(\eta)$ (gain/losses-dependent now), and harmonic oscillator potential

$$U(\xi,\eta) = \Omega^2(\eta)\xi^2 / 2$$

satisfy to the following exact integrability conditions

$$\Omega^2(\eta) D(\eta) = \frac{W(R,D)}{R^2 D} \frac{dR}{d\eta} - \frac{d}{d\eta}\left[\frac{W(R,D)}{RD} \right] = \frac{d}{d\eta} \ln D \frac{d}{d\eta} \ln R - \frac{d^2}{d\eta^2} \ln D - R \frac{d^2}{d\eta^2}\left(\frac{1}{R} \right) \quad (3)$$

where

$$W(R,D) = R\frac{dD}{d\eta} - D\frac{dR}{d\eta}$$

is the Wronskian of the functions $R(\eta)$ and $D(\eta)$.

Eq. (1) is written down here in the most general normalized form taking into account the varying dispersion $D(\eta)$, nonlinearity $R(\eta)$, gain or loss $\Gamma(\eta)$, and both linear and parabolic trapping potentials. Depending on concrete physical applications, two independent variables η and ξ can be considered as coordinate and time for temporal optical solitons; or propagation distance and transverse coordinate in the case of spatial solitons; or both time t and space x in different BEC's applications. In particular, in nonlinear waveguide optics Eq.(1) was proposed recently as the basic model for spatial similariton pulses amplification in graded-index optical waveguides[38-41]. The parameter $\sigma = \pm 1$ in (1) separates attractive (bright solitons) and repulsive (dark solitons) self-interactions. Notice that Eq.(2) belongs to the class of the exactly integrable by IST method[37,39].

All nonautonomous solitons move with varying amplitudes $\kappa(\eta)$ and speeds $v(\eta)$ given by the varying in time spectral parameter of the IST theory:

$$\Lambda(\eta) = \nu(\eta) + i\kappa(\eta) = \frac{D_0 R(\eta)}{R_0 D(\eta)} \left[\Lambda(0) + \frac{R_0}{D_0} \int_0^\eta \frac{\alpha(\tau) D(\tau)}{R(\tau)} d\tau \right], \quad (4)$$

where the initial conditions are represented by: $\Lambda(0) = \nu_0 + i\kappa_0,\ D_0 = D(0), R_0 = R(0).$

Equation (3) represents the fundamental law of the soliton adaptation to external potentials. From the physical point of view, the adaptation means that solitons remain self-similar and do not emit dispersive waves during their interactions with external potentials. In BEC's applications, where the variables $\eta = t$, $\xi = x$ and $D(t) = 1$, Eq. (2) has a quantum mechanical meaning and the soliton adaptation law Eq.(3) can be rewritten in the form

$$\Omega^2(t) = -R(t) \frac{d^2}{dt^2}\left[\frac{1}{R(t)} \right] \quad (5)$$

Eq.(5) establishes a one-to-one correspondence between varying in time nonlinearity *R(t)* and parabolic trapping potential $U(x,t)=\Omega^2(t)x^2/2$. Because of this, the interested reader can choose different control functions *R*(*t*) in Eq.(5) to find the corresponding trapping potentials.

In general, according to the soliton adaptation law, 1D experimental arrangement with nonautonomous solitons and solitonlike similaritons should be inspected to be as close as possible to the optimal map of parameters given by Eqs.(3-5), as might appear at first sight.

Let us show, on the one hand, that great care should be exercised in applications of the law Eq.(3) when due to the transformation of trapping potential from cigar-shaped to ball-shaped structure, the soliton collapse may occur. It means that the assumptions made in deriving the 1D model break down. On another hand, we stress that the soliton adaptation law Eq.(3) carries the germ of the idea of collapse suppression and 3D soliton bullets generation. We confirm this idea by direct computer experiments.

3. 3D TO 1D REDUCTIONS AND TRANSFORMATIONS: 3D SOLITONLIKE NONAUTONOMOUS BULLETS

The parallels between nonlinear guided waves phenomena in optics and nonlinear guided matter waves can be clearly demonstrated by rigorous comparison of the so-called nonlinear parabolic equation describing laser pulse dynamics in a waveguide with varying (say, square or parabolic) transverse profile of the refractive index

$$ik_0\frac{\partial}{\partial x}E(r,t)=\left[-\frac{1}{2}\nabla_\perp{}^2+\frac{1}{2}k_0k_2\frac{\partial^2}{\partial t^2}-k^2\frac{n_2}{n_0}|E(r,t)|^2-k_0{}^2\frac{n_{core}{}^2-n_0{}^2}{2n_0{}^2}U_\perp(\frac{r_\perp}{r_0})\right]E(r,t) \tag{6}$$

and the Gross-Pitaevskii equation governing dynamics of the complex wave function *Ψ(**r**,t)* of the condensate

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2++U_{3D}(\mathbf{r},t)+G_{3D}(t)|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t) \tag{7}$$

In Eq.(6), *E(**r**,t)* is the slowly varying envelope of the electrical field; $r_\perp{}^2 = y^2 + z^2$ characterizes the displacement from the laser beam center; and linear refractive index is given by: $n_{lin} = n_0[1+(n_{core} - n_0)\, U_\perp(r_\perp/r_0)/\, n_0]$, where indexes n_0 and n_{core} are used to denote the cladding and core indexes of fiber with radius r_0; dimensionless function $U_\perp(r_\perp/r_0)$ describes the refractive index cross-section profile, time is given by $t = t - x/v$; parameters $v = \partial\omega/\partial k$ and $k_2 = \partial^2 k/\partial\omega^2$ denote the group velocity of the laser pulse and its second-order dispersion.

The nonlinearity in BEC $G_{3D}(t) = 4\pi\hbar^2 a_s(t)/m$ arises from the density dependent mean field interactions between atoms proportional to the *s*-wave scattering length a_s. In Eq.(7), $r = (x,y,z)$ is the displacement from the 3D trap $U_{3D}(r,t)$ center and *m* refers to the mass of the atom. The condensate wave function is normalized to the total number of atoms $\int|\Psi(r,t)|^2 dr = N$ and the density of atoms is given by $n(r) = |\Psi(r,t)|^2$.

Whereas in nonlinear optics, 3D→1D reduction is the well-established procedure, by contrast, in the matter wave physics, the deduction of the 1D model from the 3D GPE is not so straightforward and 3D→1D reduction is not well justified for varying in time trapping potentials. In particular, let us show that some assumptions made in deriving the 1D GPE break down when trapping potential varies in time.

Let us consider a cylindrically symmetric harmonic trapping potential

$$V_{3D}(x,y,z,t)=\frac{1}{2}m\Omega_{ho\|}{}^{2}(t)x^{2}+\frac{1}{2}m\Omega_{ho\perp}{}^{2}r_{\perp}{}^{2} \tag{8}$$

with constant radial trap frequency $\Omega_{ho\perp}$ and varying in time longitudinal one $\Omega_{ho\|}(t)$

$$\Omega_{ho\|}(t)=\Omega_{ho\|}(t=0)\times\Omega(t) \tag{9}$$

where dimensionless function *Ω(t)* describes the time-varying trapping potential only in the axial direction *x*, so that a transverse trap does not depend on time $\Omega_{ho\perp}=const$.

Nonlinear dynamics of a quasi-one-dimensional (a cigar-shaped) condensate under much greater transverse than axial confinement $\Omega^{2}{}_{ho\perp}<<\Omega^{2}{}_{ho\|}$ can be considered in the framework of the GPE

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=\left(-\frac{\hbar^{2}}{2m}\nabla^{2}++U_{3D}(\mathbf{r},t)+\frac{4\pi\hbar^{2}a_{s}(t)}{m}\left|\Psi(\mathbf{r},t)\right|^{2}+\frac{1}{2}m\Omega^{2}{}_{ho\|}(t)x^{2}+\frac{1}{2}m\Omega^{2}{}_{ho\perp}(y^{2}+z^{2})\right)\Psi(\mathbf{r},t) \tag{10}$$

with factorized wave function

$$\Psi(\mathbf{r},t)=\Xi(\mathbf{r}_{\perp},t)\psi(x,t) \tag{11}$$

for transverse ($r_{\perp}{}^{2}$ = y^2 + z^2) and longitudinal (*x*) dependencies. The 3D GP equation (10) can be reduced to 1D model, if one can neglect excitations of higher order transverse modes so that the atoms occupy only the ground state of their transverse motion. The wave function separation (11) can be achieved formally by a two time scale expansion with slow response time scale $T_{trap||}$ and fast response time scale $T_{trap_{\perp}}$ given by

$$\begin{aligned} T_{trap_{\|}}&=\frac{2\hbar}{m\Omega^{2}{}_{ho\|}r_{0}{}^{2}}=\frac{2a^{2}{}_{ho\|}}{m\Omega_{ho\|}r_{0}{}^{2}}=T_{ho\|}\frac{a^{2}{}_{ho\|}}{\pi r_{0}{}^{2}}\\ T_{trap_{\perp}}&=\frac{2\hbar}{m\Omega^{2}{}_{ho\perp}r_{0}{}^{2}}=\frac{2a^{2}{}_{ho\perp|}}{m\Omega_{ho\perp}r_{0}{}^{2}}=T_{ho\perp}\frac{a^{2}{}_{ho\perp}}{\pi r_{0}{}^{2}} \end{aligned} \tag{12}$$

Although, in a strict sense, this separation can be done only in the linear problem, the effective one-dimensional nonlinear coefficient appears to be a factor in the nonlinear term resulting from the subsequent transverse integration:

$$G_{1D}=\frac{\iint G_{3D}(t)\Xi^{4}(y,z)dydz}{\iint\Xi^{2}(y,z)dydz}. \tag{13}$$

The averaging procedure Eq. (13), in full analogy with the same procedure for optical solitons in fibers[31-35], leads to $G_{1D}(t)=2\hbar^{2}a_{s}(t)/\left(ma_{ho\perp}{}^{2}\right)$, where $a_{ho\perp}$ is the transverse harmonic oscillator length. Physically, transition to the quasi-1D description is possible if the change of the chemical potential due to the mean-field interaction is much smaller than the level spacing in the transverse trapping potential.

The net result is the 1D Gross-Pitaevskii equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{2\hbar^2 a_s(t)}{ma^2_{ho\perp}}\left|\psi(\mathbf{r},t)\right|^2+\frac{1}{2}m\Omega^2{}_{ho\|}(t)x^2\right)\psi \qquad (14)$$

where $\psi(x,t)$ is the quasi one-dimensional wave function normalized to the number of particles. This equations belogs to the class of nonautonomous NLSE with so-called nonautonomous soliton solutions[36-60],

It is easy to understand that as a consequence of varying-in-time nonlinearity and trapping potential, the atom cloud is subjected to wide axial variations, including its self-compression, and, in general, it may be that the initially quasi-1D cigar-shaped structure of the atom cloud transforms continuously to the ball-shaped 3D structure. That is why the reverse process of 1D-to-3D transformation from cigar-shaped to ball-shaped geometry can be observed because of the soliton self-compression effect. This is crucial for the purposes of matter-wave soliton management under the exact integrability condition.

By using the following transformations of functions and coordinates: $\psi(\xi,\eta)\to\sqrt{a_{ho\|}/N}\,q(x,t)$; $\eta\to\Omega_{ho\|}t$; $\xi\to x/a_{ho\|}$; and $G(\eta)\to\varepsilon a_s(t)a_{ho\|}{}^2/a_{ho\perp}{}^2$, where $\varepsilon=2a_s(0)N/a_{ho\|}$; $a_{ho\perp},a_{ho\|}$ are the transverse and longitudinal harmonic oscillator lengths, Eq. (14) can be rewritten in the most general form represented by master Eq.(1) taking into account the varying dispersion $D(\eta)$, nonlinearity $R(\eta)$, gain or loss $\Gamma(\eta)$, and both linear $2\alpha(\eta)\xi$ and parabolic $\Omega^2(\eta)\xi^2/2$ trapping potentials.

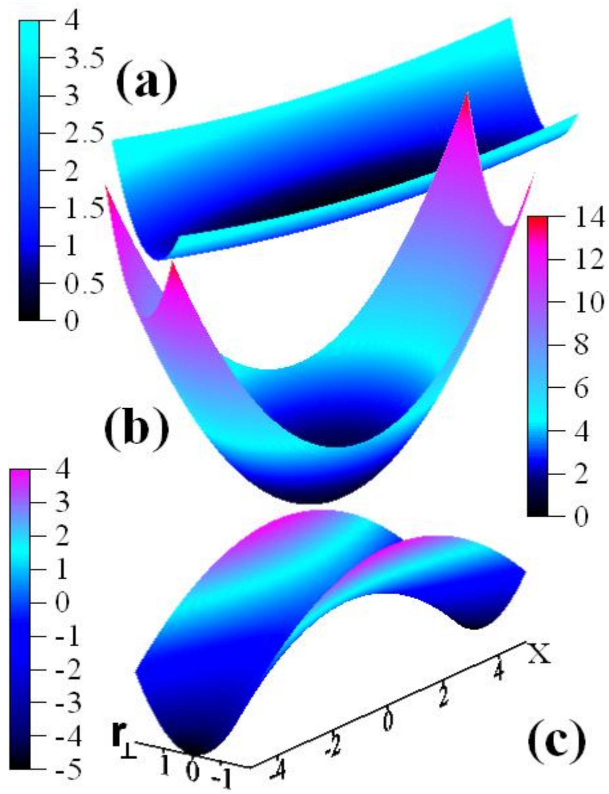

Fig.1. 1D to 3D transformations from cigar-shaped to ball-shaped geometry of trapping potential $V(r,t)=\delta^2\Omega^2(t)x^2/2+r_\perp^{\,2}/2$ with the aspect ratio $\delta^2=\Omega^2_{ho\|}/\Omega^2_{ho\perp}$ under the exact integrability conditions given by Eq. (5): (a) $\Omega^2(t)=R(t)=1$; (b) $\Omega^2(t)=-2\gamma/(1+\gamma t^2)$ and $R(t)=(1+\gamma t^2)^{-1}$ with negative γ = -0.1 at t = 3.0; (c) $\Omega^2(t)=\beta\varpi^2\cos\varpi t/(1+\beta\cos\varpi t)$, $R(t)=(1+\beta\cos\varpi t)^{-1}$ after choosing β = 0.95, ϖ = 1.0 and t = π.

Conventionally, 1D NLSE model works well for cigar-shaped atom cloud when its axial width is much larger than the diameter of the "cigar" (see Fig. 1(a)). The crucial questions, though, are: what happens if both the axial width of cigar-shaped trapping potential and nonlinearity inside the atom cloud vary with time? And, that is more important, what happens if parabolic potential $U(x,t)=\Omega^2(t)x^2/2$ periodically transforms from the trapping ($\Omega^2(t)>0$) axial potential to the repulsive ($\Omega^2(t)<0$) one, as is shown in Figs.1(b) and 1(c). It is easy to understand that as a consequence of varying in time nonlinearity and trapping potential, it may be that initially quasi-1D-cigar-shaped structure of the atom cloud transforms to the ball-shaped 3D structure and under the law of soliton adaptation given by Eq.(3) (see Fig.1). In Fig.1 we present an extremely strong 1D to 3D transformation of trapping parabolic potential according the law Eq.3, where the nonlinearity is chosen as $R(t)=1/(1+0.1t^2)$. It will be shown later a typical soliton self-compression scenario of the initially cigar-shaped density of the atom cloud, its necking and slitting when 1D approximation breaks up. It is this effect that disrupts the solitonlike structure of the atom cloud and leads to the soliton collapse.

For better visualization of these ideas, we present here three examples of such reverse and nontrivial 1D-to-3D transformations (see Figures 1–4 for details). Figure 1 shows the trapping harmonic potential of the exactly integrable nonautonomous system given by

$$R(t) = \frac{1}{1+\gamma t^2}; \qquad \Omega^2(t) = -\frac{2\gamma}{1+\gamma t^2} \tag{15}$$

To be specific, we show in Figures 2 and 3 all control functions for this case, so that Figs. 2-3 are probably the most dramatic direct manifestations of the reverse 1D-to-3D transformations of a trapping potential.

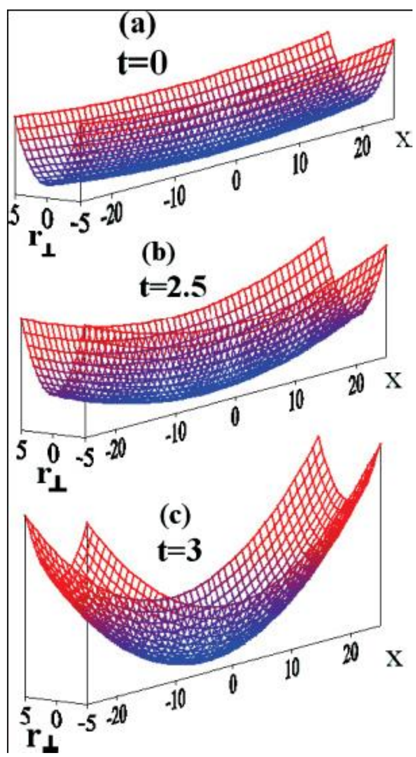

Fig.2. The reverse process of 1D-to-3D transformation from cigar-shaped geometry to ball-shaped geometry of trapping potential $V(x,t) = \frac{1}{2}\kappa^2\Omega^2(t)x^2 + \frac{1}{2}r_\perp{}^2$ under the exact integrability conditions given by Eq. (15) with negative $\gamma = -0.1$ at different time scales. (a) t = 0, (b) t = 2.5, (c) t = 3.0, after choosing the aspect ratio $\kappa^2 = 0.04$ ($r^2 = y^2 + z^2$).

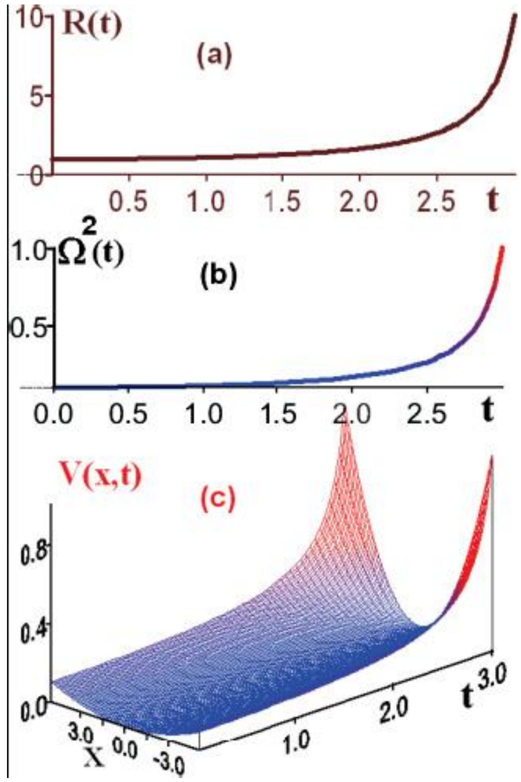

Fig.3. Main soliton-management functions (a) $R(t)$, (b) $\Omega^2(t)$, and (c) axial trapping potential varying in time under the exact integrability conditions given by Eq. (15) after choosing the aspect ratio $\kappa^2 = 0.04$ of trapping potential and $\gamma = -0.1$.

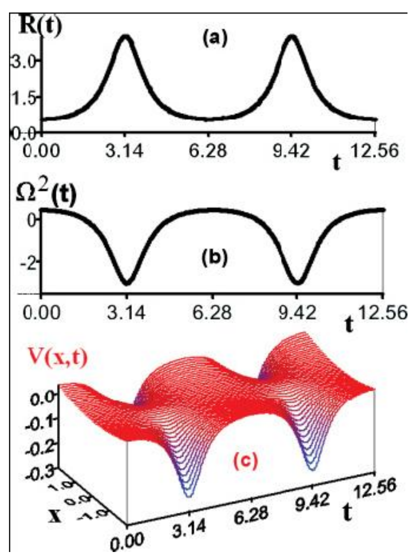

Fig.4. Main soliton-management functions (a-b), and (c) axial trapping potential varying in time under the exact integrability conditions given by Eqs. (16) after choosing the aspect ratio $\kappa^2 = 0.04$ of trapping potential $V(x,t) = \frac{1}{2}\kappa^2\Omega^2(t)x^2 + \frac{1}{2}r_\perp{}^2$.

Figure 4 shows an example of periodic sign-reversible variations in time of the confining potential (periodic-in-time repulsive and attractive parabolic potentials), which have been calculated according to the exact integrability conditions given by Eqs. (16)

$$\Gamma(t) = \gamma_2 \varpi_2 \cos \varpi_2 t; \quad R(t) = \exp[\gamma_2 \sin \varpi_2 t]; \quad \Omega^2(t) = -\gamma_2 \varpi_2{}^2 \left[\sin \varpi_2 t + \gamma_2 \cos^2(\varpi_2 t)\right] \tag{16}$$

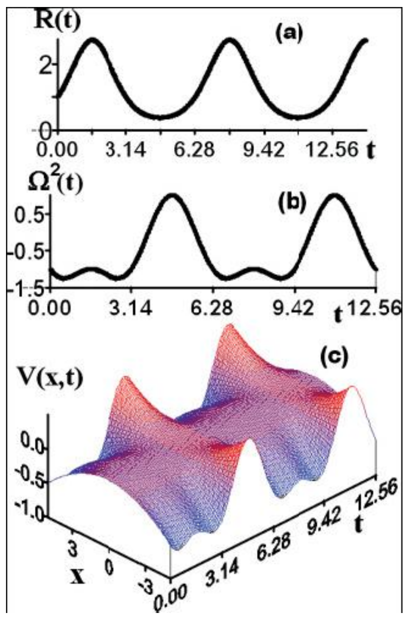

Fig.5. Main soliton management functions (a), (b), and (c) axial trapping potential varying in time under the exact integrability conditions given by Eqs. (17).

And, finally, Fig. 5 represents the results of periodic 1D-to-3D transformations from attractive cigar-shaped to repulsive ball-shaped structures that have been calculated in the framework of the exact integrability conditions given by

$$R(t) = \frac{1}{1 + \beta \cos \varpi t}; \qquad V(x,t) = \frac{1}{2}\Omega^2(t) x^2 = \frac{1}{2}\frac{\beta \varpi^2 \cos \varpi t}{1 + \beta \cos \varpi t} x^2. \tag{17}$$

It should be emphasized that all soliton solutions presented here are strictly valid for the limiting case of 1D condensate. That is why it is natural to ask to what extent it is possible to consider the general solution for the condensate density $n(\mathbf{r}, t)$ in the factorized form,

$$\frac{n(\mathbf{r},t)}{n_0} = \frac{|\Psi(\mathbf{r},t)|^2}{|\Psi(\mathbf{r}=0,t=0)|^2} = \frac{|1-\Phi(0)|^2}{|1-\Phi(t)|} \sec h^2 \left\{ \frac{x}{L_{sol\|}} \frac{|1-\Phi(0)|}{|1-\Phi(t)|} \right\} \exp\left(-\frac{y^2+z^2}{L_\perp^{\ 2}} \right), \qquad (18)$$

where $n(\mathbf{r},t)/n_0$ is the density of atoms at the point in question normalized to its initial peak value $n_0 = N/(2\pi L_\perp^{\ 2} L_{sol\|})$, and $L_\perp$ and $L_{sol\|}$ are the initial radial and axial widths of the wave function.

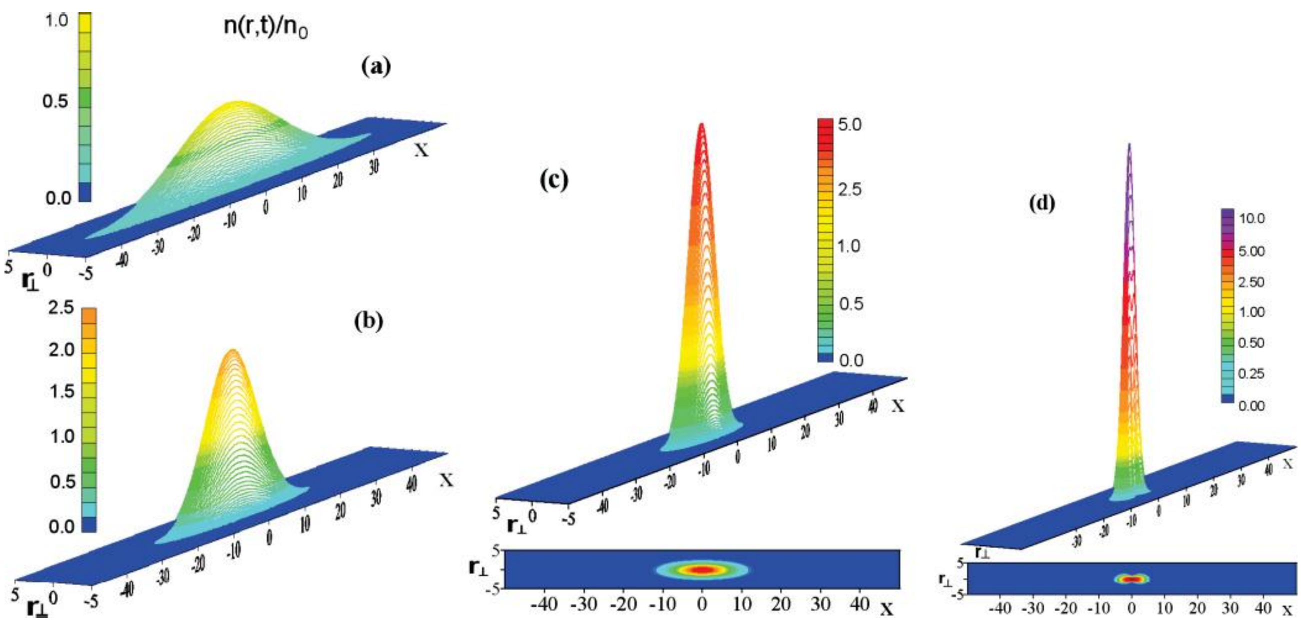

Fig.6. Self-compression of matter-wave soliton near the Feshbach resonance calculated within the framework of the nonautonomous GPE model (10) after choosing the soliton management functions (8) with negative $\gamma = -0.1$ at different dimensional time scales normalized to Ti_{nt} = 30 ms. (a) $-t = 0$, (b) $-t = 2.25$, (c) $-t = 2.75$, and (d) $-t = 3.0$, which correspond to (b) 67.5 ms, (c) 82.5 ms, and (d) 90 ms. The initial state represents a quasi-1D cigar-shaped cloud of condensed $N = 10^3$ atoms with axial and radial sizes Lsol|| = 15 μm and $L_\perp$ = 1.5 μm, respectively. In all figures the condensate density $n(\mathbf{r}, t)$ is normalized to the initial peak value $n_0 = 0.5 \times 10^{13}$ cm^{-3}. Axial and radial widths are given in microns.

In Eq. (18), the exact 1D soliton solution has been chosen for the longitudinal direction, whereas in the transverse direction a Gaussian ansatz is the optimal one. To reveal the basic properties of 3D matter-wave solitary waves, we have solved numerically the nonautonomous GP equation [Eq. (10)] by applying an operator-splitting method based on 3D fast Fourier transform (FFT) which is a well-developed computational method in nonlinear optics, acoustics, and self-focusing theory. The condition of validity of 1D soliton self-compression has been tested by solving Eq. (10) with initial conditions given by Eq. (18) and taking as the managing functions several profiles given by Eqs. (3).

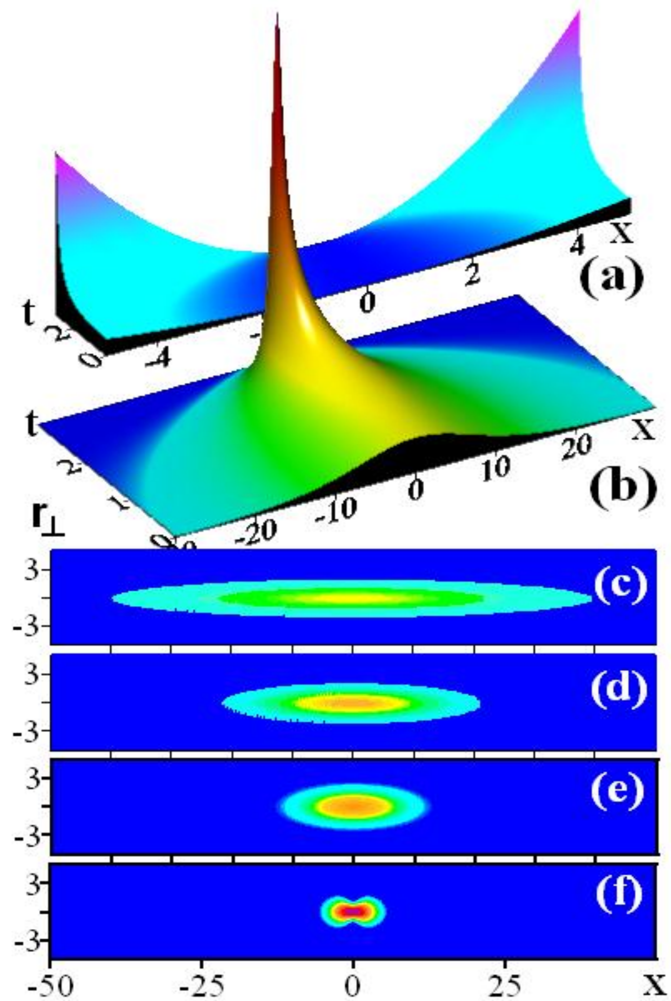

Fig.7. (a) The axial trapping potential transformation with time, and (b) soliton self-compression effects calculated within the framework of the GPE model Eq.(10) after choosing $\Omega^2(t) = 2\gamma/(1-\gamma t^2)$. The contour plots (c-f) of the atomic density of initially quasi-1*D*-cigar-shaped atom cloud of condensed N=10^3 atoms for: (c) t = 0; (d) t = 2.25; (e) t = 2.75; and (f) t = 3.0. In all plots the condensate density $n(r,t)$ is normalized to the initial peak value n_0 = 0.5×10^{13}cm^{-3}. Axial and radial widths are given in units of microns.

In order to simulate an extremely strong 1D-to-3D transformation of the initially cigar-shaped cloud of atoms into the BEC bullet, the varying-in-time nonlinearity $R(t)$ was chosen as $R(t) = 1/(1 - 0.1t^2)$ in accordance with Eqs. (3-5). Initial state presents a cigar-shaped density of condensed $N = 10^3$ atoms with axial and radial sizes Lsol = 15 μm and $L_\perp$ = 1.5 μm, respectively, and with initial 3D peak density n_0 = 0.5 × 10^{13} cm^{-3}. Axial and radial trapping harmonic potential frequencies have been chosen as $\Omega_{ho\perp}$ = $2\pi \times 52$ Hz and $\Omega_{ho\|}$ = $2\pi \times$ 0.52 Hz, respectively, and initial scattering length was chosen as $a_s\ (t = 0) = -2a_0$ to simulate its increasing up to ten times *as* (t = $t$fin) = −20a0 and to calculate corresponding self-compression of the atom cloud by a factor of ten.

The main results of computer simulations are presented in Fig. 6 at different dimensionless time scales: (a) $-t$ = 0, (b) $-t$ = 2.25, (c) $-t$ = 2.75, and (d) $-t$ = 3.0, which are normalized on Tint = 30 ms and correspond to 67.5 ms (b), 82.5 ms (c), and 90 ms (d) in Fig. 6.

In Figs. 6(a)–6(c), the condensate density $n(\mathbf{r}, t)$ is normalized to the initial peak density value n0 = 0.5×10^{13} cm^{-3}. Direct computer experimentation enables us to draw the following conclusions. First, our simulations confirm the quadratic in time soliton compression scenario at the initial stage of self-compression. Second, the essential finding in our simulations is that during the self-compression, the elongated soliton separates on two ball-shaped solitary-like structures. In Fig. 6, we show both a typical soliton self-compression scenario and the initial stage of soliton necking and splitting near the resonance time when the peak density rises dramatically with time [see Fig. 6(d)]. It is this effect that disrupts the solitonic structure of the atom cloud and breaks the effect of soliton adaptation to the external potential given by the exact integrability conditions [Eqs. (3)]. As a matter of fact, based on computer simulations, we conclude that a 1D nonautonomous GPE model accurately describes the matter-wave soliton self-compression until the axial $L_{sol\|}$ and radial $L_\perp$ widths of a soliton satisfy the following "empirical" condition: $L_{sol\|}{}^2(t) \geq 3L_\perp$. Because of this, the main assumption based on the factorized form Eq. (18) holds true only within the limits $L_{sol\|}{}^2(t) \geq 3L_\perp$. By using the simplest approximation for a 1D soliton width in the quasi-1D BEC with varying-in-time scattering length,

$$L_{sol\|}{}^2(t) n_{1D}(t) \approx \frac{L_\perp{}^2}{4|a_s(t)|}, \qquad (19)$$

where $n_{1D}(t) = N/L_{sol||}(t)$ represents the 1D density of the condensate, we can find the following "empirical" condition of the validity of the 1D nonautonomous soliton concept, namely, the 1D nonautonomous matter-wave soliton exists under exact integrability conditions if and only if varying-in-time scattering length $as(t)$ and 1D density $n_{1D}(t)$ satisfy the following inequality:

$$|a_s(t)| n_{1D}(t) \leq 1/12 \cong 0.1.$$

(20)

To date, bright matter-wave solitons and trains of bright solitons have been studied in experiments with Bosonic lithium having attractive interactions. The negative value of the scattering length for ^{7}Li prevents the formation of a condensate with more then a few thousand atoms. If we consider the typical experimental conditions for the generation of matter-wave solitary waves in quasi-1D ^{7}Li condensate with typical radial width $a_{ho||}$ = 1.0

μm, we conclude that 1D approximation is still valid when the width of a compressed BEC soliton is of the order of Lsol|| = 1.7 μm (see Fig.7).

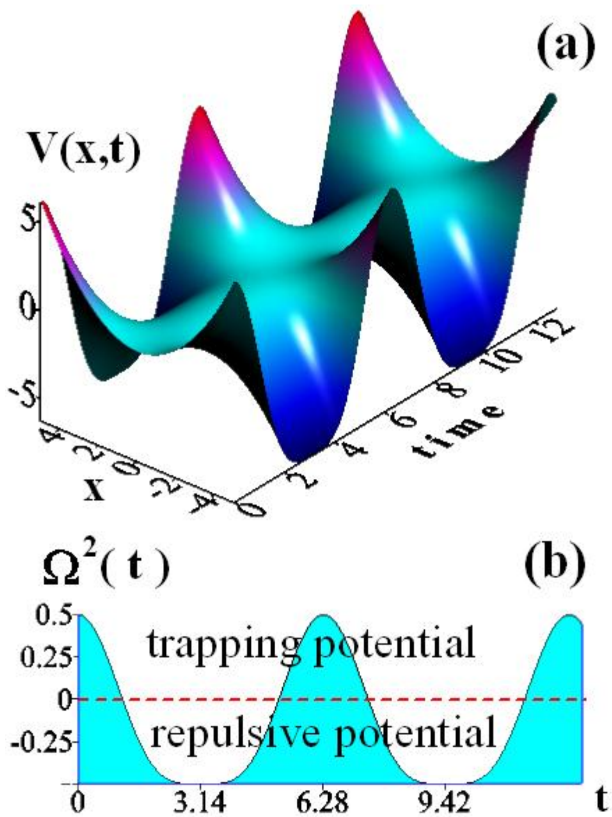

Fig.8. Nonautonomous soliton bullets generation under the exact integrability conditions $\Omega^2(t)=\beta\varpi^2\cos\varpi t/(1+\beta\cos\varpi t)$, $R(t)=(1+\beta\cos\varpi t)^{-1}$ after choosing β = 0.95, ϖ = 1.0 : (a) periodic management of the nonlinearity; (b) periodic opposite in phase sign-reversal oscillations of the external harmonic oscillator potential; (c) periodic reversal transformation of quasi-1D-cigar-shaped atom cloud to ball-shaped bullet-like atom cloud.

It should be emphasized that Eq.(3) carries the germ of the idea of the 3D soliton collapse suppression (see Figure 8). This idea is based on the periodic sign reversal changes of the trapping potential which are opposite in phases to periodic variations of the nonlinearity so that these two processes are alternating to each other and the peaks of nonlinearity inside the atomic cloud coincide in time with repulsive ball-shaped structures of harmonic potential shown in Figures 8 and 9.

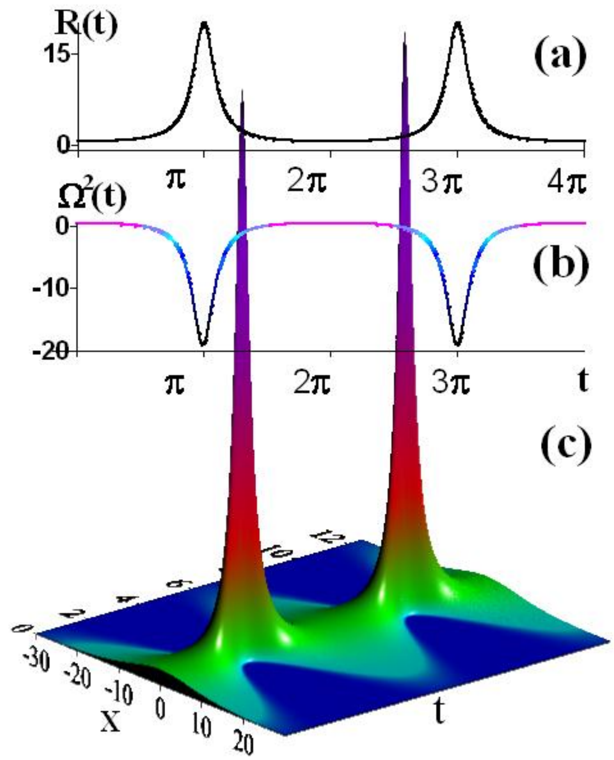

Fig.9. Nonautonomous soliton bullets generation under the exact integrability conditions $\Omega^2(t)=\beta\varpi^2\cos\varpi t/(1+\beta\cos\varpi t)$, $R(t)=(1+\beta\cos\varpi t)^{-1}$ after choosing β = 0.95, ϖ = 1.0 : (a) periodic management of the nonlinearity; (b) periodic opposite in phase sign-reversal oscillations of the external harmonic oscillator potential; (c) periodic reversal transformation of quasi-1*D*-cigar-shaped atom cloud to ball-shaped bullet-like atom cloud.

To be specific, we show in Fig.9 all control functions $R(t)$ and $\Omega(t)$ for this case which have been calculated according to the exact integrability conditions given by Eq.(3) where the strength of nonlinearity $R(t)=1/(1+\beta\cos\varpi t)$ varies with ϖ=16π rad/s and period T = 125 ms. To reveal the basic properties of 3D matter wave solitary bullets, we have solved numerically the nonautonomous Gross-Pitaevskii equation Eq.(10) with varying in time sign-reversal harmonic oscillator potential by applying operator splitting method based on 3D fast Fourier transform. The initial state presents a cigar-shaped density of condensed $N = 10^3$ atoms with axial and radial sizes $L_{sol||}$ = 15μm and $L_{\perp}$ = 1.5μm and with initial 3D peak density n_0 = 0.5×10^{13}cm^{-3}. Axial and radial trapping harmonic potential frequencies have been chosen as $\Omega_{ho\perp} = 2\pi\times52$ Hz and $\Omega_{ho||} = 2\pi\times0.52$ Hz, initial scattering length was chosen as a_s(t=0) = -2a_0 (in units of the Bohr radius a_0 = 0.0529 nm).

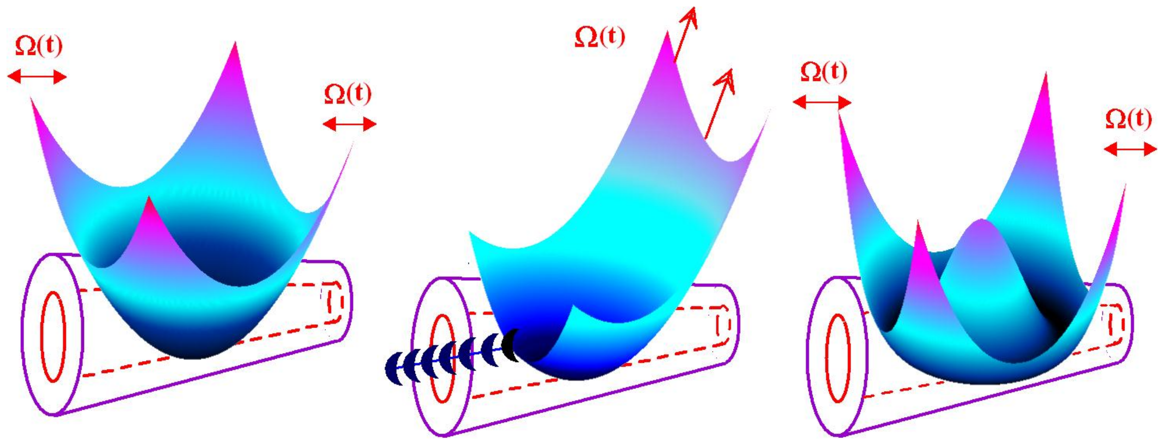

Fig.9. Different experimental setups for nonautonomous soliton bullets generation.

Different experimental setups for nonautonomous soliton bullets generation in BEC are illustrated in Fig.10. We stress that in nonlinear optical applications, periodic sign-reversal variations of trapping potential mean periodic redistributions of spatial dependences of the refractive indexes, for example, in a periodic graded-index nonlinear structure with alternating wave-guiding and anti-wave-guiding segments. These structures are known from the first days of nonlinear fiber optics[33]. In particular, using the parameter regions defined in[38-41] for silicon graded-index planar waveguide and for AlGaAs slab waveguide, we conclude that the required input power levels are about 100

4. CONCLUSION

The intrinsic similarities between the mean-field GPE model and the NLSE model imply the existence of common fundamental nonlinear phenomena, independent of the physical origin of the nonlinearity. In this sense, matter wave solitons are similar to optical solitons in fibers and, therefore, the parallels with optical solitons in dispersion decreasing and oscillating fibers with varying nonlinearity and gain or loss provides novel experimental opportunities not only for optical soliton physics but for the BEC physics as well.

We have focused on the most physically important situations where the applied magnetic field is varying in time linearly and periodically. We have shown that in nonlinear optical applications, periodic graded-index nonlinear structures with alternating wave-guiding and anti-wave-guiding segments open remarkable opportunities in studies of BEC systems by performing experiments in nonlinear optical systems.

The interpenetration of the main ideas and methods being used in different fields of science and technology becomes at present one of the decisive factors for the progress of science as a whole. Among the most spectacular examples of such an interchange of ideas and theoretical methods for analysis of various physical phenomena is the problem of solitary wave formation in the framework of the nonlinear Schrödinger equation models with linear and harmonic oscillator potentials. This model is used in a variety of fields of modern nonlinear science and probably will be able to play a basic role similar to that played in due time by the model of a quantum mechanical linear harmonic oscillator in the development of modern physics.

We conclude by saying that the concept of adaptation is of primary importance in nature, and the law of soliton adaptation to an external potential offers many opportunities for future scientific studies.

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